LOCAL LIMIT THEOREM FOR SYMMETRIC RANDOM WALKS IN GROMOV-HYPERBOLIC GROUPS

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ABSTRACT. Completing a strategy of Gouëzel and Lalley [GL11], we prove a local limit theorem for the random walk generated by any symmetric finitely supported probability measure on a non-elementary Gromov-hyperbolic group: denoting by R the inverse of the spectral radius of the random walk, the probability to return to the identity at time n behaves like $CR^{-n}n^{-3/2}$. An important step in the proof is to extend Ancona's results on the Martin boundary up to the spectral radius: we show that the Martin boundary for R-harmonic functions coincides with the geometric boundary of the group. In an appendix, we explain how the symmetry assumption of the measure can be dispensed with for surface groups.

1. INTRODUCTION

Consider a countable group Γ (with identity denoted by e), together with a probability measure μ whose support generates Γ as a semigroup (we say that μ is admissible). Multiplying random elements of Γ distributed independently according to μ, one obtains a random walk on Γ. The local limit problem consists in determining good asymptotics for the transition probabilities $p_n(x, y)$ of this random walk. Let us assume for simplicity that μ is finitely supported. For $\Gamma = \mathbb{Z}^d$, simple Fourier computations show that $p_n(e, e) \sim Cn^{-d/2}$ if the walk is centered, and $p_n(e, e) \sim CR^{-n}n^{-d/2}$ for some $R > 1$ if the walk is not centered. Similar asymptotics hold in nilpotent groups by the deep results of Varopoulos and Alexopoulos [Ale02].

When the group is not amenable, $p_n(e, e)$ decays exponentially fast. The situation is well understood for semisimple Lie groups and absolutely continuous measures since the work of Bougerol [Bou81]: the probability to return to a fixed neighborhood of the identity behaves like $CR^{-n}n^{-a}$ for some $R > 1$ (depending on the measure one considers) and some $a > 1$ only depending on the geometry of the group. In the simplest case of rank one groups, $a = 3/2$. It is reasonable to conjecture that similar asymptotics (with the same a) hold for random walks on cocompact lattices of such semisimple Lie groups, but the proofs of Bougerol (based on representation theory) do not adapt well, and this question is essentially open.

A notable exception is the case of free groups: in this situation, the generating function of the transition probabilities (also called the Green function) $G_r(x, y) = \sum r^n p_n(x, y)$ is an algebraic function of r. A careful study of its first singularity then yields the asymptotics

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of $p_n(x,y)$. For free groups, this is due to Lalley [Lal93], and the asymptotics is of the form $p_n(x,y) \sim C(x,y)R^{-n}n^{-3/2}$, in accordance with the results of Bougerol in rank one Lie groups. Most free products can also be treated similarly, see [Woe00, Chapter III] and references therein.

Recently, together with Lalley, we were able to treat in [GL11] some non-amenable groups where the Green function is not expected to be algebraic. We proved that, for a cocompact lattice of $\mathrm{PSL}(2,\mathbb{R})$, and for a finitely supported symmetric measure μ, the above asymptotics $p_n(x,y) \sim C(x,y)R^{-n}n^{-3/2}$ still holds. Henceforth, we will refer to this situation as the $\mathrm{PSL}(2,\mathbb{R})$-case. The overall strategy can in fact be formulated in any Gromov-hyperbolic group (including in particular all cocompact lattices in rank one semisimple Lie groups), but a crucial point in the proof really relies on two-dimensional geometry. In this article, we provide a completely different argument for this crucial point, making it possible to extend the results of [GL11] to any Gromov-hyperbolic group.

We say that the walk is *aperiodic* if there exists an odd integer n such that $p_n(e,e) > 0$. In this case, $p_n(e,e) > 0$ for all large enough n.

Theorem 1.1. *Let Γ be a finitely generated non-elementary Gromov-hyperbolic group. Let μ be an admissible finitely supported symmetric probability measure on Γ. Denote by $R > 1$ the inverse of the spectral radius of the corresponding random walk. For any $x, y \in \Gamma$, there exists $C(x,y) > 0$ such that*

$$p_n(x,y) \sim C(x,y)R^{-n}n^{-3/2}$$

if the walk is aperiodic. If the walk is periodic, this asymptotics holds for even (resp. odd) n if the distance from x to y is even (resp. odd).

The proof of the analogous theorem in the $\mathrm{PSL}(2,\mathbb{R})$-case in [GL11] is divided in three steps, as follows:

(1) One shows that Ancona's results [Anc87] on the Martin boundary extend up to $r = R$. In particular, the Martin kernel $K_{r,\xi}(x) = G_r(x,\xi)/G_r(e,\xi)$ converges when ξ tends to a point in the geometric boundary of Γ, uniformly in $r \in [1,R]$.
(2) Using the Cannon automaton coding geodesics in the group, and thermodynamic formalism in the resulting subshift of finite type, one gets estimates for the sums $\sum_{x\in\Gamma} G_r(e,x)G_r(x,e)$ when $r \to R$ in terms of a pressure function. This implies that $r \mapsto G_r(e,e)$ almost satisfies a differential equation. Asymptotics of this function follow.
(3) From the asymptotics of $G_r(e,e)$, one deduces the asymptotics of $p_n(e,e)$ using tauberian theorems (and a little bit of spectral theory). The asymptotics of $p_n(x,y)$ are proved in the same way.

From this point on, this article is subdivided into three sections, each devoted to one of those three steps. We will give further comments, explain quickly the arguments in [GL11], and insist on the differences between the $\mathrm{PSL}(2,\mathbb{R})$-case and the general case of Gromov-hyperbolic groups. The main difference is in the first step: the proof of [GL11] is deeply 2-dimensional, and the general argument is completely different. For the second step, a significant technical complication appears: in the $\mathrm{PSL}(2,\mathbb{R})$-case, the Cannon automaton (a combinatorial object coding the geodesics in the group) is transitive, while this is not the case in general. To overcome this difficulty, we use additional information from the first step, and a technique of Calegari and Fujiwara [CF10]. Finally, the third step is exactly the same in the $\mathrm{PSL}(2,\mathbb{R})$-case or in the general case, we will only give some details for the convenience of the reader.

While we have tried to make this article as self contained as possible, [GL11] provides a good introduction to some concepts and techniques that we use. The letter C denotes a constant that may vary from line to line. Since most arguments work exactly in the same way for symmetric or nonsymmetric measures, we have written most proofs without using the assumption of symmetry. It only plays a role in the proof of Lemma 2.6 (the central lemma to obtain Ancona inequalities) and in Section 4. We expect that the first step (Ancona inequalities) should be true without any symmetry assumption on the measure. While we are not able to prove it in general, we are able to obtain it for cocompact discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. The identification of the Martin boundary at the spectral radius follows. The argument is given in Appendix A.

2. Ancona inequalities up to the spectral radius

2.1. **The Green function.** Consider an admissible finitely supported probability measure μ on a countable group Γ. It defines a random walk on Γ. Let $R = R(\mu) = \limsup p_n(e,e)^{-1/n}$ (when μ is symmetric, this is the inverse of the spectral radius of the Markov operator associated to the random walk on ℓ^2). The Green function is defined for $1 \leqslant r < R$ and $x, y \in \Gamma$ by $G_r(x,y) = \sum r^n p_n(x,y)$. By a result of Guivarc'h, it is convergent even for $r = R$ if the group carries no recurrent random walk (this is in particular true for non-amenable groups). One should think of $G_r(x,y)$ as the average number of passages in y if the random walk starts from x, but for the measure $r\mu$ instead of μ. In particular, for larger r, G_r gives more weight to longer paths.

If $\gamma = (x, x_1, \dots, x_{n-1}, y)$ is a path of length n from x to y, its r-weight $w_r(\gamma)$ is $r^n \prod_{i=0}^{n-1} p(x_i, x_{i+1})$ (where $x_0 = x$ and $x_n = y$ by convention, and we write $p(a,b) = \mu(a^{-1}b)$ for the probability to jump from a to b). By definition, $G_r(x,y) = \sum w_r(\gamma)$, where the sum is over all paths from x to y.

If Ω is a subset of Γ, one defines the restricted Green function $G_r(x,y;\Omega)$ as $\sum w_r(\gamma)$ where the sum is over all paths $\gamma = (x, x_1, \dots, x_{n-1}, y)$ such that $x_i \in \Omega$ for $1 \leqslant i \leqslant n-1$. If A is a subset of Γ such that any trajectory of the random walk from x to y has to go through A, one has

$$G_r(x,y) = \sum_{a\in A} G_r(x,a;A^c)G_r(a,y) = \sum_{a\in A} G_r(x,a)G_r(a,y;A^c), \tag{2.1}$$

where A^c denotes the complement of A. Indeed, the first (resp. second) formula is proved by splitting a path from x to y according to its first (resp. last) visit to A. More generally, if Ω is a subset of Γ containing x and y, the above formula holds restricted to Ω, i.e.,

$$G_r(x,y;\Omega) = \sum_{a\in A\cap\Omega} G_r(x,a;A^c\cap\Omega)G_r(a,y;\Omega) = \sum_{a\in A\cap\Omega} G_r(x,a;\Omega)G_r(a,y;A^c\cap\Omega).$$

Assuming that Γ is finitely generated, we can consider a word distance d on Γ coming from a finite symmetric generating set. If x and y are at distance d, there is a path from x to y with probability bounded from below by C^{-d}, and staying close to a geodesic segment from x to y. We deduce that, for any z,

$$C^{-d(x,y)} \leqslant G_r(x,z)/G_r(y,z) \leqslant C^{d(x,y)}, \tag{2.2}$$

and similar inequalities hold for the Green functions restricted to any set containing a fixed size neighborhood of a geodesic segment from x to y. These inequalities are called Harnack inequalities.

The first visit Green function is $F_r(x,y) = G_r(x,y;\{y\}^c)$. It only takes into account the first visits to y. For $r = 1$, this is the probability to reach y starting from x. One has $G_r(x,y) = F_r(x,y)G_r(y,y) = F_r(x,y)G_r(e,e)$, by the formula (2.1) for $A = \{y\}$. Moreover, $F_r(x,y)G_r(y,z) \leqslant G_r(x,z)$ (since the concatenation of a path from x to y with a path from y to z gives a path from x to z). Dividing by $G_r(e,e)$, one gets

$$F_r(x,y)F_r(y,z) \leqslant F_r(x,z). \tag{2.3}$$

We also obtain

$$G_r(x,y)G_r(y,z) \leqslant G_R(e,e)G_r(x,z). \tag{2.4}$$

The Martin boundary is the set of pointwise limits of sequences of functions $K_{r,y_n}(x) = G_r(x,y_n)/G_r(e,y_n)$ when y_n tends to infinity. Since these functions are normalized by $K_{r,y}(e) = 1$, and r-harmonic except at y, limits exist, are nonzero, and r-harmonic everywhere (since the measure μ has finite support). Understanding the Martin boundary amounts to understanding for which sequences y_n the functions K_{r,y_n} converge.

The derivative of G_r with respect to r can be computed. Indeed,

$$(rG_r(x,y))' = \sum_{z\in\Gamma} G_r(x,z)G_r(z,y),$$

where the prime indicates the derivative with respect to r. This equation for $x = y = e$ shows that

$$\eta(r) := \sum_z G_r(e,z)G_r(z,e) < +\infty \quad \text{for all } r < R. \tag{2.5}$$

Let us introduce a convenient notation: we shall write

$$H_r(x,y) = G_r(x,y)G_r(y,x). \tag{2.6}$$

In the symmetric case, this is simply the square of the Green function. With this notation, $\eta(r) = \sum_{z\in\Gamma} H_r(e,z)$. Since G_r satisfies (2.4), H_r also satisfies this inequality.

2.2. **Ancona inequalities.** Consider a finitely generated group Γ. Its Cayley graph is endowed with the word metric coming from any finite set of generators. One says that Γ is Gromov-hyperbolic (or simply hyperbolic) if there exists δ such that any geodesic triangle in this Cayley graph is δ-thin, i.e., each side of the triangle is contained in the δ-neighborhood of the union of the two other sides. This notion is invariant under quasi-isometry, and therefore independent of the choice of the generators (see [GdlH90] for more details on hyperbolic groups). The geometric intuition to have is that any finite set of points in an hyperbolic group is isometric to a finite set of points in a tree, up to some constant only depending on the number of points. In particular, statements regarding the relative positions of points can be reduced to statements in trees, that are easy to check combinatorially. This intuition is made precise by the following theorem ([GdlH90, Theorem 2.12]).

Theorem 2.1. *For any $n \in \mathbb{N}$ and $\delta > 0$, there exists a constant $C = C(n, \delta)$ with the following property. Consider a subset A of a δ-hyperbolic space of cardinality at most n. There exists a map Φ from F to a metric tree such that, for any $x, y \in A$,*

$$d(x, y) - C \leqslant d(\Phi(x), \Phi(y)) \leqslant d(x, y).$$

An hyperbolic group Γ (or more generally any geodesic Gromov-hyperbolic space) has a well defined geometric boundary $\partial\Gamma$: this is the set of semi-infinite geodesics, where two such geodesics are identified if they stay a bounded distance away. This boundary is a compact space, and $\Gamma \cup \partial\Gamma$ is also compact.

Consider now an admissible finitely supported probability measure μ on a non-elementary hyperbolic group Γ (i.e., not quasi-isometric to $\{0\}$ or $\mathbb{Z}$). Let $R = R(\mu)$, it is strictly larger than 1 since Γ is not amenable. Ancona proved in [Anc87] that, for any $r < R$, the Martin boundary for r-harmonic functions coincides with the geometric boundary: K_{r,y_n} converges pointwise if and only if y_n converges to a point $\xi \in \partial\Gamma$, and the limits are different for different points of the boundary.

A crucial inequality in Ancona's proof is the fact that the converse inequality to (2.4) holds for any $r < R$ whenever y is close to a geodesic from x to z (with a constant a priori depending on r). In other words, typical trajectories from x to z follow the geodesic sufficiently well so that they are likely to pass close to y. When r increases, G_r gives more and more weight to long trajectories, that are more likely to go further from the geodesic. Hence, this Ancona estimate is more and more subtle when r increases. Proving such an estimate for $r = R$ is a crucial step in the proof of Theorem 1.1.

Definition 2.2. *A probability measure μ on a Gromov-hyperbolic group Γ satisfies uniform Ancona inequalities if there exists a constant $C > 0$ such that, for any $x, z \in \Gamma$ and for any y close to a geodesic segment from x to z, for any $r \in [1, R(\mu)]$,*

$$G_r(x, z) \leqslant C G_r(x, y) G_r(y, z).$$

Theorem 2.3. *If μ is admissible, finitely supported and symmetric on a non-elementary Gromov-hyperbolic group, it satisfies uniform Ancona inequalities.*

This result has been proved in [GL11] for cocompact lattices of $\mathrm{PSL}(2, \mathbb{R})$, using very specific two-dimensional arguments. The main idea in the new argument to follow is to combine a supermultiplicativity estimate (originating in [DPPS11] for counting problems) with a geometric construction of random barriers in hyperbolic space. We will write $|x|$ for the distance of x to the identity e (for some fixed word distance), and $\mathbb{S}_k$ for the sphere of radius k around e. The rest of this section is devoted to the proof of Theorem 2.3. We fix a non-elementary Gromov-hyperbolic group Γ and an admissible probability measure μ. We do not assume yet that μ is symmetric, since it will only be important in Lemma 2.6 below.

Lemma 2.4. *There exists $C > 0$ such that, for any $x, y \in \Gamma$, there exists $a \in \Gamma$ of length at most C such that $|xay| \geqslant |x| + |y|$.*

Proof. Fix $C_0 = C(4, \delta) > 0$ such that any configuration of at most 4 points can be approximated by a tree with error at most C_0, as in Theorem 2.1.

We will rely on the classical construction of free groups with two generators in Γ as follows. An hyperbolic element of Γ is an element u of Γ such that the left-multiplication by

u has two fixed points at infinity, an attracting one and a repelling one, denoted by u_+ and u_-. Consider two hyperbolic elements u, v in Γ such that the four points u_+, u_-, v_+, v_- are distinct (this is possible since Γ is non-elementary, see the proof of [GdlH90, Theorem 8.37]), and fix small disjoint neighborhoods $V(u_+), V(u_-), V(v_+), V(v_-)$ of those points in $\Gamma \cup \partial\Gamma$. If N is large enough, any x in the complement of $V(a_+)$ (for $a \in F = \{u, u^{-1}, v, v^{-1}\}$) shares only a short beginning with a^N. More precisely, there exists $K > 0$ independent of N such that, in a tree approximation Φ of e, x, a^N with error at most C_0, the branches from $\Phi(e)$ leading to $\Phi(x)$ and $\Phi(a^N)$ split before time K. Increasing N, we can also assume that $|a^N| \geqslant 4K + 3C_0$ for all $a \in F$.

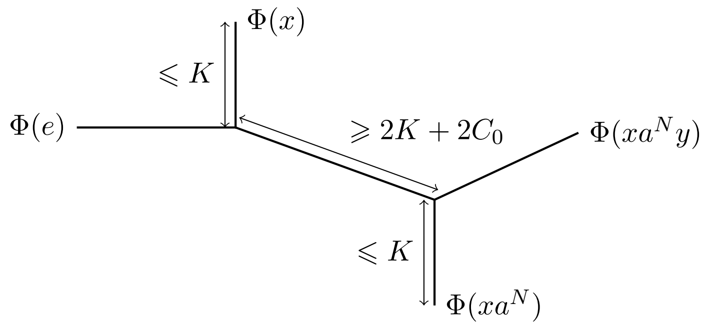


FIGURE 1. Approximating tree for $e, x, xa^N, xa^N y$.

Consider now two points $x, y \in \Gamma$. One can choose $a \in F$ such that $x^{-1} \notin V(a_+)$ and $y \notin V(a_-)$. Consider a tree approximation $\Phi : \{e, x, xa^N, xa^N y\} \to T$. The geodesic paths from $\Phi(x)$ to, respectively, $\Phi(e)$ and $\Phi(xa^N)$, split before time K by construction since $x^{-1} \notin V(a_+)$. In the same way, the paths from $\Phi(xa^N)$ to, respectively, $\Phi(x)$ and $\Phi(xa^N y)$ also split before time K since $y \notin V(a_-)$ (see Figure 1). Hence,

$$\begin{aligned}
|xa^N y| &\geqslant d(\Phi(e), \Phi(xa^N y)) \\
&\geqslant d(\Phi(e), \Phi(x)) + d(\Phi(x), \Phi(xa^N)) + d(\Phi(xa^N), \Phi(xa^N y)) - 2 \cdot 2K \\
&\geqslant |x| - C_0 + |a^N| - C_0 + |y| - C_0 - 4K \\
&\geqslant |x| + |y| .
\end{aligned}$$

□

We recall the notation $H_r(x, y) = G_r(x, y)G_r(y, x)$ from (2.6).

Lemma 2.5. *There exists $C > 0$ such that, for any $k \in \mathbb{N}$, $\sum_{x \in \mathbb{S}_k} H_R(e, x) \leqslant C$.*

Proof. Fix $r < R$. Write $u_k(r) = \sum_{x \in \mathbb{S}_k} H_r(e, x)$. To $x \in \mathbb{S}_k$ and $y \in \mathbb{S}_\ell$ one can associate thanks to the previous lemma a point $\Phi(x, y) = xay \in \bigcup_{k+\ell \leqslant i \leqslant k+\ell+C} \mathbb{S}_i$. By (2.4), we have

$$\begin{aligned}
H_r(e, x)H_r(e, y) &\leqslant CH_r(e, x)H_r(e, a)H_r(e, y) = CH_r(e, x)H_r(x, xa)H_r(xa, xay) \\
&\leqslant CH_r(e, xay).
\end{aligned}$$

Let us estimate the number of preimages under Φ of some point z. Let γ_z be a geodesic segment from e to z. If $z = xay$ and x is far away from γ_z, a tree approximation shows that $|z|$ is significantly smaller than $|x| + |a| + |y|$. This is impossible by construction. Therefore,

x is contained in a ball of fixed radius $B(\gamma_z(k), C)$. In particular, the number of possibilities for x is uniformly bounded. Arguing in the same way for y, we deduce that, for some $C > 0$, each point has at most C preimages under Φ.

Finally,

$$\begin{aligned} u_k(r)u_\ell(r) &= \sum_{x\in\mathbb{S}_k, y\in\mathbb{S}_\ell} H_r(e,x)H_r(e,y) \leqslant C \sum_{x\in\mathbb{S}_k, y\in\mathbb{S}_\ell} H_r(e,\Phi(x,y)) \\ &\leqslant C \sum_{i=k+\ell}^{k+\ell+C} \sum_{z\in\mathbb{S}_i} H_r(e,z) \leqslant C \sum_{i=k+\ell}^{k+\ell+C} u_i(r). \end{aligned}$$

As $r < R$, the sum $\sum_{x\in\Gamma} H_r(e,x)$ is finite by (2.5). In particular, the sequence $u_k(r)$ is summable, and reaches its maximum $M(r)$ at some index $k_0(r)$. Using the previous equation with $k = \ell = k_0(r)$, we get $M(r)^2 \leqslant C(C+1)M(r)$, hence $M(r) \leqslant C(C+1) = D$.

Finally, for every $r < R$, for every $k \in \mathbb{N}$, one has $\sum_{x\in\mathbb{S}_k} H_r(e,x) \leqslant D$. The lemma follows by letting r tend to R. □

The following lemma is the main estimate in the proof of Theorem 2.3. It gives super-exponentially small estimates for the R-probabilities of paths staying too far away from geodesics, implying that such paths are very unlikely and will not contribute a lot to G_R.

Lemma 2.6. *Assume that μ is finitely supported and symmetric. There exist $n_0 > 0$ and $\varepsilon > 0$ such that, for any $n \geqslant n_0$, for any $x, y, z \in \Gamma$ on a geodesic segment (in this order) with $d(x,y) \geqslant n$ and $d(y,z) \geqslant n$,*

$$G_R(x,z; B(y,n)^c) \leqslant 2^{-e^{\varepsilon n}}.$$

Proof. Without loss of generality, one can assume $y = e$.

Fix some $\varepsilon > 0$ very small, and let $N = \lfloor e^{\varepsilon n} \rfloor$. In this proof, we will write C for a generic constant independent of ε. The idea of the proof is to construct N *barriers* $A_1, \dots, A_N$ such that any trajectory of the random walk going from x to z outside of $B(e,n)$ has to go through A_1, then A_2, and so on. Decomposing a trajectory according to its first visit to A_1, then A_2, and so on, we obtain as in (2.1)

(2.7)
$$G_R(x,z;B(e,n)^c) \leqslant \sum_{a_1\in A_1} \cdots \sum_{a_N\in A_N} G_R(x,a_1)G_R(a_1,a_2)\cdots G_R(a_{N-1},a_N)G_R(a_N,z).$$

We will construct the barriers so that, writing $A_0 = \{x\}$ and $A_{N+1} = \{z\}$, one has for any $0 \leqslant i \leqslant N$

$$\sum_{a\in A_i} \sum_{b\in A_{i+1}} G_R(a,b)^2 \leqslant 1/4. \tag{2.8}$$

This implies the desired estimate on $G_R(x,z;B(y,n)^c)$ by Cauchy-Schwarz, as follows. To write it formally, it is more convenient to express things in terms of operators, as in [Led11]. Define an operator $L_i : \ell^2(A_{i+1}) \to \ell^2(A_i)$ by $L_i f(a) = \sum_{b\in A_{i+1}} G_R(a,b)f(b)$. The sum to estimate in (2.7) is $(L_0 \cdots L_N \delta_z)(x)$, it is therefore bounded by $\prod \|L_i\|$. Moreover,

$$\|L_i f\|_{\ell^2}^2 = \sum_{a\in A_i} \left| \sum_{b\in A_{i+1}} G_R(a,b) f(b) \right|^2 \leqslant \sum_{a\in A_i} \left(\sum_{b\in A_{i+1}} G_R(a,b)^2 \right) \cdot \left(\sum_{b\in A_{i+1}} |f(b)|^2 \right)$$
$$= \left(\sum_{a\in A_i} \sum_{b\in A_{i+1}} G_R(a,b)^2 \right) \|f\|_{\ell^2}^2 .$$

With (2.8), we obtain $\|L_i\| \leqslant \left(\sum_{a\in A_i} \sum_{b\in A_{i+1}} G_R(a,b)^2 \right)^{1/2} \leqslant 1/2$, and the result of the lemma follows.

It remains to construct barriers satisfying (2.8). The construction is geometric, and is done in the hyperbolic space $\mathbb{H}^m$ for some $m \geqslant 2$ (or rather its model as the euclidean unit ball in $\mathbb{R}^m$, with the boundary at infinity identified with the unit sphere S^{m-1} in $\mathbb{R}^m$). By [BS00], the group Γ with its word metric is roughly similar to a subset of such a space: if m is large enough, there exists a mapping $\Psi : \Gamma \to \mathbb{H}^m$ and $\lambda > 0$, $C > 0$ such that $|\lambda d_{\mathbb{H}}(\Psi(u), \Psi(v)) - d(u,v)| \leqslant C$ for all $u, v \in \Gamma$. The image under Ψ of a geodesic in Γ is a quasi-geodesic in $\mathbb{H}^m$, therefore it remains uniformly close to a true hyperbolic geodesic (see for instance [GdlH90, Theorem 5.11]). It follows that it does not make a serious difference to use geodesics in Γ or in $\mathbb{H}^m$.

The hyperbolic geodesic from $\Psi(x)$ to $\Psi(z)$ can be extended biinfinitely. Composing with an hyperbolic isometry, we can assume that this geodesic goes through the center O of the ball model of $\mathbb{H}^m$, and that $\Psi(e)$ is a bounded distance away from O. Let ξ be the endpoint of this hyperbolic geodesic in negative time. To an angle $\theta \in [0, \pi]$, we associate the union of all the semiinfinite geodesics $[O\zeta)$ (with $\zeta \in S^{m-1}$) making an angle θ with $[O\xi)$ (its boundary at infinity is the circle of points at distance θ of ξ in S^{m-1}). Let then $A(\theta)$ be the set of points a in $B(e,n)^c \subset \Gamma$ such that $\Psi(a)$ is at a distance at most C_0 of such a geodesic. If C_0 is chosen large enough, a path of the random walk going from x to z in $B(e,n)^c$ can not jump over $A(\theta)$ since μ has finite support, so that $A(\theta)$ is a barrier.

In $X = [0, \pi]$, consider $X_i = [(2i-1)/N, 2i/N]$ (for $1 \leqslant i \leqslant N$). Those intervals are separated by $1/N \sim e^{-\varepsilon n}$. In each of them, we will choose an angle θ_i and let $A_i = A(\theta_i)$. One should then ensure that (2.8) is satisfied. To do so, we will choose each θ_i at random as follows. Let $\Omega = \prod_{i=1}^N X_i$, endowed with the product of the probability measures $N\,\mathrm{dLeb}$ on X_i. Define a function f_i on Ω by

$$f_i(\theta_1, \dots, \theta_N) = \sum_{a\in A(\theta_i), b\in A(\theta_{i+1})} G_R(a,b)^2,$$

where by convention $A(\theta_0) = \{x\}$ and $A(\theta_{N+1}) = \{z\}$. One should find a value of $\overline{\theta} = (\theta_1, \dots, \theta_N)$ such that $f_i(\overline{\theta}) \leqslant 1/4$ for all i. We will show that

$$\int f_i \leqslant C e^{-\rho n}, \tag{2.9}$$

for some $\rho > 0$ independent of ε. It follows that $\int (\sum f_i) \leqslant C(1+N)e^{-\rho n} \leqslant C(1+e^{\varepsilon n})e^{-\rho n}$. Choosing ε small enough, this is exponentially small, and is in particular bounded by $1/4$

for large enough n. This yields a point $\overline{\theta}$ with $\sum f_i(\overline{\theta}) \leqslant 1/4$, for which the corresponding barriers satisfy (2.8).

Let us now prove (2.9). We will only give the argument for $1 \leqslant i \leqslant N-1$: the case of f_0 and f_N is slightly different (since $A_0 = \{x\}$ and $A_{N+1} = \{z\}$ are fixed), it turns out to be analogous to the general case, but simpler. Fix some $i \in [1, N-1]$. To each $a \in \Gamma$ and $j \in \{i, i+1\}$, we associate the set $X_j(a)$ of angles θ in X_j such that $a \in A_j(\theta)$. By definition,

$$\int f_i = \sum_{a,b\in\Gamma} G_R(a,b)^2 \cdot N \operatorname{Leb}(X_i(a)) \cdot N \operatorname{Leb}(X_{i+1}(b)).$$

For $a \in \mathbb{S}_k$, its image under Ψ is at distance at least αk of O in $\mathbb{H}^m$, for some $\alpha > 0$. If one moves away from this point by at most C_0, the visual angle from O varies by at most $Ce^{-\alpha k}$. It follows that $\operatorname{Leb}(X_j(a)) \leqslant Ce^{-\alpha k}$. Since $N \leqslant e^{\varepsilon n}$, we obtain

$$\int f_i \leqslant Ce^{2\varepsilon n} \sum_{a,b} G_R(a,b)^2 e^{-\alpha|a|} e^{-\alpha|b|},$$

where the sum is restricted to those a and b outside of $B(e,n)$ and whose images under Ψ belong to the C_0-neighborhoods of the sectors delimited respectively by X_i and X_{i+1}. Writing $u = a^{-1}b$ (with $|u| \leqslant |a| + |b|$), we get

$$\int f_i \leqslant Ce^{2\varepsilon n} \sum_{u} G_R(e,u)^2 e^{-\alpha|u|} N(u),$$

where $N(u)$ is the number of ways to decompose u as $a^{-1}b$. Fix a point u, and such a decomposition $u = a^{-1}b$.

The hyperbolic geodesics from O to, respectively, $\Psi(a)$ and $\Psi(b)$, make an angle at least $e^{-\varepsilon n}/2$. Therefore, they are far away from each other outside of the ball $B(O, 2\varepsilon n)$. It follows from a tree approximation that

$$d_{\mathbb{H}}(\Psi(a), \Psi(b)) \geqslant |\Psi(a)| + |\Psi(b)| - 4\varepsilon n - C.$$

Since Ψ is a quasisimilarity, we deduce that $|u| = d(a,b) \geqslant |a| + |b| - C\varepsilon n - C$. In particular, if ε is small enough, since $|a| \geqslant n$ and $|b| \geqslant n$, we obtain $|u| \geqslant n$. It also follows from this argument that a geodesic in the group from a to b has to pass through the ball $B(e, C\varepsilon n)$, since geodesics in the group and in hyperbolic space remain a bounded distance away. Let γ be a geodesic segment from e to u in Γ, then $a\gamma$ is a geodesic segment from a to b. There exists a time j such that $a\gamma(j) \in B(e, C\varepsilon n)$. Finally, $a \in \bigcup_{j=0}^{|u|} \gamma(j)^{-1} B(e, C\varepsilon n)$, which gives at most $(|u|+1)C^{C\varepsilon n}$ possibilities for a. Arguing similarly for b, be obtain $N(u) \leqslant (|u|+1)^2 e^{C\varepsilon n}$ for some $C > 0$.

Finally, we have

$$\int f_i \leqslant Ce^{2\varepsilon n} \sum_{|u|\geqslant n} G_R(e,u)^2 e^{-\alpha|u|} (|u|+1)^2 e^{C\varepsilon n} \leqslant Ce^{(C+2)\varepsilon n} \sum_{|u|\geqslant n} G_R(e,u)^2 e^{-\alpha|u|/2}.$$

Since $\sum_{|u|=k} G_R(e,u)^2$ equals $\sum_{|u|=k} H_R(e,u)$ by symmetry of μ, it is uniformly bounded by Lemma 2.5. Therefore, $\int f_i$ is bounded by $Ce^{C\varepsilon n} e^{-\alpha n/2}$. If ε is small enough, this is at most $Ce^{-\alpha n/4}$. This proves (2.9) and concludes the proof of the lemma. $\square$

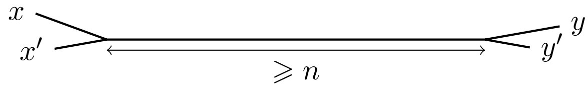
The following lemma is proved in [GL11], and is elementary (see the proof of Theorem 4.1 there):

Lemma 2.7. *Let μ be an admissible measure on a Gromov-hyperbolic group. Assume that, for all $K > 0$, there exists n_0 such that, for all $n \geqslant n_0$, for all points x, y, z on a geodesic segment (in this order) with $d(x, y) \in [n, 100n]$ and $d(y, z) \in [n, 100n]$, one has $G_R(x, z; B(y, n)^c) \leqslant K^{-n}$. Then μ satisfies uniform Ancona inequalities. It even satisfies strong uniform Ancona inequalities (as defined below in Definition 2.8).*

To prove this lemma, one uses recursively its assumptions to show that most r-weight is concentrated on paths staying close enough to the geodesic from x to z, and in particular passing in a ball of fixed radius around y. This lemma, together with Lemma 2.6, proves uniform Ancona inequalities for symmetric measures, i.e., Theorem 2.3. Strong uniform Ancona inequalities (see below) are then deduced as in done [GL11, Theorem 4.6].

We need the following strengthening of Ancona inequalities:

Definition 2.8. *A measure μ on a Gromov-hyperbolic group satisfies strong uniform Ancona inequalities if it satisfies uniform Ancona inequalities and, additionally, there exist constants $C > 0$ and $\rho > 0$ such that, for all points x, x', y, y' whose configuration is approximated by a tree as follows*



for any $r \in [1, R]$,

$$\left| \frac{G_r(x, y)/G_r(x', y)}{G_r(x, y')/G_r(x', y')} - 1 \right| \leqslant Ce^{-\rho n}. \tag{2.10}$$

Ancona inequalities ensure that the quantity $\frac{G_r(x,y)/G_r(x',y)}{G_r(x,y')/G_r(x',y')}$ in the definition is bounded from above and from below. Strong Ancona inequalities ensure that this quantity is exponentially close to 1 in terms of the distance between the sets of points $\{x, x'\}$ and $\{y, y'\}$. These bounds are not formal consequences of Ancona inequalities, but they are consequences of Ancona inequalities in suitable domains (that follow from Lemma 2.6). Applying Lemmas 2.6 and 2.7, we obtain the following result, strengthening Theorem 2.3.

Theorem 2.9. *If μ is admissible, finitely supported and symmetric on a non-elementary Gromov-hyperbolic group, it satisfies strong uniform Ancona inequalities.*

When one takes $x' = e$, then the quantity appearing in (2.10) is the ratio $K_{r,y}(x)/K_{r,y'}(x,)$ of the Martin kernels. The theorem implies that, when y_i tends to a point $\xi \in \partial\Gamma$, the sequence $K_{r,y_i}(x)$ is a Cauchy sequence (since the points x, e and y_i, y_j satisfy the assumptions of the definition with a large n for large enough i, j). Hence, it converges to a function $K_{r,\xi}(x)$. This is the main step in the proof that the Martin boundary for r-harmonic functions coincides with the geometric boundary (one should also check that $K_{r,\xi} \neq K_{r,\eta}$ for $\xi \neq \eta$, which is easy). We omit the (classical) details, see for instance [INO08].

3. Asymptotics of the Green function

Let Γ be a non-elementary Gromov-hyperbolic group. Our goal in this section is to prove the following theorem.

Theorem 3.1. *Let μ be an admissible probability measure on Γ satisfying strong uniform Ancona inequalities. For any $x, y \in \Gamma$, there exists $C(x, y) > 0$ such that, when r tends to $R = R(\mu)$,*

$$\partial G_r(x, y)/\partial r \sim C(x, y)(R - r)^{-1/2}.$$

Throughout this section, we fix a measure μ satisfying the assumptions of this theorem. Theorem 2.9 shows that it is the case for finitely supported symmetric measures, but symmetry will play no additional role in this section. We will concentrate mainly on the proof of Theorem 3.1 for $x = y = e$, since the general case will follow easily.

In a sense, the proof of Theorem 3.1 is essentially done in [GL11], but there is an important technical difference: a (well chosen) Markov automaton for a surface group is transitive, while there can be several components in a general hyperbolic group. This means that, in the thermodynamic formalism, we will have to deal with several dominating components. This problem is solved thanks to a technique of Calegari and Fujiwara [CF10] and to Lemma 2.5. This sketch of the argument might be sufficient for experts, but since there are several technical subtleties we will give most details below. There is a significant overlap with some arguments in [GL11], but this seems necessary to keep the argument understandable. Two significant differences with [GL11] (in addition to the existence of several dominating components in the automaton, and directly related to this issue) are that we need some a priori estimates (proved in Subsection 3.1), and that for $r < R$ we will associate to G_r a measure living on the group, not on the boundary.

3.1. **A priori estimates.** The main idea behind the proof of Theorem 3.1, as in [GL11], is that the function $G_r(e, e)$ almost satisfies a differential equation. By (2.1), its derivative with respect to r is essentially $\sum_x G_r(e, x)G_r(x, e) = \sum_x H_r(e, x)$ (recall the notation (2.6)), and its second derivative is essentially $\sum_{x,y} G_r(e, y)G_r(y, x)G_r(x, e)$. To prove that $G_r(e, e)$ almost satisfies a differential equation, we should relate those quantities. The next proposition gives such a (crude) relation.

Proposition 3.2. *There exists $C > 0$ such that, for all $r \in [1, R)$,*

$$C^{-1} \leqslant \frac{\sum_{x,y} G_r(e, y)G_r(y, x)G_r(x, e)}{\left(\sum_x G_r(e, x)G_r(x, e)\right)^3} \leqslant C.$$

Proof. Consider two points x, y. The triangle with vertices e, x, y is thin, so there exists a point w (defined uniquely up to a finite set) which is close to each of its sides. By the Ancona inequality, we have $G_r(e, y) = G_r(w^{-1}, w^{-1}y) \leqslant CG_r(w^{-1}, e)G_r(e, w^{-1}y)$, since a geodesic segment from w^{-1} to $w^{-1}y$ passes close to e by construction. Similar estimates hold along $[y, x]$ and $[x, e]$, and we obtain

$$\begin{aligned} &G_r(e, y)G_r(y, x)G_r(x, e) \\ &\qquad \leqslant CG_r(w^{-1}, e)G_r(e, w^{-1}) \cdot G_r(w^{-1}x, e)G_r(e, w^{-1}x) \cdot G_r(w^{-1}y, e)G_r(e, w^{-1}y). \end{aligned}$$

The points $w^{-1}, w^{-1}x$ and $w^{-1}y$ determine x and y. Using the notation H_r and summing over x and y, we get

$$\sum_{x,y} G_r(e,y)G_r(y,x)G_r(x,e) \leqslant C \sum_{a,b,c} H_r(e,a)H_r(e,b)H_r(e,c).$$

This is one of the inequalities of the proposition.

For the reverse inequality, for any $u \in \Gamma$, write $B(u)$ for the set of points x such that a geodesic from e to x passes close to u. Lemma 2.4 ensures that, for any $z \in \Gamma$, there exists a uniformly bounded a such that $uaz \in B(u)$. In particular, Harnack inequalities (2.2) give $G_r(e,z) \leqslant C(u)G_r(e,uaz)$ (and H_r satisfies the same inequality). Hence, $\sum_{z\in\Gamma} H_r(e,z) \leqslant C(u) \sum_{v\in B(u)} H_r(e,v)$. Choose now three geodesic segments γ_1, γ_2 and γ_3 (with endpoints denoted by u_1, u_2, u_3), long enough and going in three different directions (this is possible since the group is non-elementary) so that the sets $B(u_i)$ are pairwise disjoint, and so that a geodesic from $B(u_i)$ to $B(u_j)$ $(i \neq j)$ has to pass close to e. We get

$$\left(\sum_z H_r(e,z)\right)^3 \leqslant C(u_1,u_2,u_3) \sum_{v_i \in B(u_i)} H_r(e,v_1)H_r(e,v_2)H_r(e,v_3).$$

By (2.4), we have $G_r(v_1,e)G_r(e,v_2) \leqslant CG_r(v_1,v_2)$, and similarly for circular permutations. This sum is therefore bounded by $\sum G_r(v_1,v_2)G_r(v_2,v_3)G_r(v_3,v_1)$. Let $y = v_1^{-1}v_2$ and $x = v_1^{-1}v_3$. The point close to the three sides of a geodesic triangle with vertices e, x and y is close to v_1^{-1} by construction, hence x and y determine v_1 (and then v_2 and v_3) up to a finite number of possibilities. We finally get

$$\left(\sum_z H_r(e,z)\right)^3 \leqslant C \sum_{x,y} G_r(e,y)G_r(y,x)G_r(x,e),$$

proving the other inequality of the lemma. □

Corollary 3.3. *There exist $A \geqslant 0$ and $C > 0$ such that, for all $r \in [1, R)$,*

$$\frac{C^{-1}}{\sqrt{A + (R-r)}} \leqslant \sum_{x\in\Gamma} H_r(e,x) \leqslant \frac{C}{\sqrt{A+(R-r)}}.$$

Moreover, $A = 0$ if and only if $\sum_{x\in\Gamma} H_R(e,x) = +\infty$.

Proof. Let $\eta(r) = \sum_x H_r(e,x)$, and $F(r) = r^2\eta(r)$. By (2.1),

$$F'(r) = 2r \sum_{x,y} G_r(e,y)G_r(y,x)G_r(x,e).$$

Therefore, Proposition (3.2) shows that $2F'(r)/F(r)^3 = (-1/F(r)^2)'$ is bounded from above and below. Integrating this estimate on an interval $[r,s]$, we get

$$C^{-1}(s-r) \leqslant 1/F(r)^2 - 1/F(s)^2 \leqslant C(s-r).$$

When s increases to R, $F(s)$ converges either to a positive constant or to infinity. Hence, $1/F(s)^2$ converges to $A \in [0,\infty)$. We get $A + C^{-1}(R-r) \leqslant 1/F(r)^2 \leqslant A + C(R-r)$, from which the corollary follows. □

We shall see later that A is in fact equal to 0. Therefore, this corollary gives the right order of magnitude $1/\sqrt{R-r}$ for the function $\eta(r) = \sum_{x\in\Gamma} H_r(e,x)$. However, to obtain Theorem 3.1, we need to get *asymptotics*, of the form $\eta(r) \sim C/\sqrt{R-r}$. The strategy will be the same, relying on the differential equation, but we will need to improve Proposition 3.2, to get convergence instead of mere bounds. This is most conveniently done using the transfer operator on a Markov automaton, as we will explain in the next subsection. Before doing this, let us state a final technical lemma, that relies on Corollary 3.3 and will be important later on.

Lemma 3.4. *Fix $a \in \Gamma$. There exists $C > 0$ such that, for any $x \in \Gamma$, for any $r \in [1, R]$,*

$$\left|\log\left(\frac{G_r(e,x)}{G_r(a,x)}\right) - \log\left(\frac{G_R(e,x)}{G_R(a,x)}\right)\right| \leqslant C\sqrt{R-r}$$

and

$$\left|\log\left(\frac{G_r(x,e)}{G_r(x,a)}\right) - \log\left(\frac{G_R(x,e)}{G_R(x,a)}\right)\right| \leqslant C\sqrt{R-r}.$$

Proof. The second estimate of the lemma can be deduced from the first one applied to the measure $\check{\mu}(g) = \mu(g^{-1})$, we will therefore concentrate on the first one.

Fix some $x \in \Gamma$. Let $f(r) = \log(G_r(e,x)/G_r(a,x))$, we will show that its derivative is bounded in absolute value by $C/\sqrt{A+R-r}$, where C is a constant that does not depend on x (of course, it may depend on a). By integration, this gives $|f(r) - f(R)| \leqslant C(\sqrt{A+R-r} - \sqrt{A})$, which is bounded by $C\sqrt{R-r}$ as desired.

We write $f(r) = \log(rG_r(e,x)) - \log(rG_r(a,x))$. With the formula (2.1) for the derivative of rG_r, we get

$$\begin{aligned} f'(r) &= \frac{\sum_y G_r(e,y)G_r(y,x)}{rG_r(e,x)} - \frac{\sum_y G_r(a,y)G_r(y,x)}{rG_r(a,x)} \\ &= \frac{1}{r}\sum_y \left(1 - \frac{G_r(a,y)/G_r(e,y)}{G_r(a,x)/G_r(e,x)}\right)\frac{G_r(e,y)G_r(y,x)}{G_r(e,x)}. \end{aligned}$$

Consider a geodesic segment γ from e to x and write $\gamma(n)$ $(0 \leqslant n \leqslant |x|)$ for the point on γ at distance n of e. Let Γ_n denote the set of points $y \in \Gamma$ whose projection on γ is $\gamma(n)$, i.e., $d(y,\gamma(n)) \leqslant d(y,\gamma(i))$ for $i \neq n$. Note that there can be several such projections – in this argument, the multiplicity is not important, otherwise one can avoid it by using only the first projection. For $y \in \Gamma_n$, the points e, a and x, y are in the configuration of Theorem 2.9, with a separating distance at least $n - C$ (for some C only depending on a). Applying this theorem, we obtain

$$|f'(r)| \leqslant C\sum_{n=0}^{|x|}\sum_{y\in\Gamma_n} e^{-\rho n}\frac{G_r(e,y)G_r(y,x)}{G_r(e,x)}.$$

For $y \in \Gamma_n$, geodesics from e to y and from y to x pass close to $\gamma(n)$. Hence, Ancona inequalities give

$$G_r(e,y)G_r(y,x) \leqslant CG_r(e,\gamma(n))G_r(\gamma(n),y)G_r(y,\gamma(n))G_r(\gamma(n),x) \leqslant CG_r(e,x)H_r(\gamma(n),y).$$

Finally,

$$\left|f'(r)\right| \leqslant C \sum_{n=0}^{|x|} \sum_{y \in \Gamma_n} e^{-\rho n} H_r(\gamma(n), y) \leqslant C \sum_{n=0}^{|x|} e^{-\rho n} \sum_{y \in \Gamma} H_r(e, \gamma(n)^{-1} y).$$

Since $\sum_{z \in \Gamma} H_r(e,z) \leqslant C/\sqrt{A+R-r}$ by Corollary 3.3 and $e^{-\rho n}$ is summable, this proves the lemma. □

3.2. **Symbolic dynamics.** For a nice introduction to the topics of this paragraph and the next one, see [CF10].

Let S be a finite symmetric generating set of the group Γ. A rooted S-labeled automaton (or simply automaton) is a finite directed graph $\mathcal{A} = (V, E, s_*)$ with distinguished vertex s_* ("start"), and a labeling $\alpha : E \to S$ of edges by generators of the group.

A *path* in the graph is a sequence of edges $e_0, \dots, e_{m-1}$ such that the endpoint of e_i is the starting point of e_{i+1}. To such a path γ, one can associate a path $\alpha(\gamma)$ in the Cayley graph of Γ by multiplying successively the generators read along the edges of the path. Let $\alpha_*(\gamma)$ be the endpoint of $\alpha(\gamma)$.

Definition 3.5. *An automaton is a strongly Markov automatic structure for Γ if:*

(1) *Every vertex $v \in V$ is accessible from the start state s_*.*
(2) *For every path γ, the path $\alpha(\gamma)$ is a geodesic path in Γ.*
(3) *The endpoint mapping α_* induced by α is a bijection of the set of paths starting at s_* onto Γ.*

In particular, the sphere $\mathbb{S}_k$ of Γ is in bijection with the set of paths of length k starting from s_*.

Every Gromov-hyperbolic group admits such a strongly Markov automatic structure, by a theorem of Cannon [Can84]. Let us fix once and for all such an automaton. An infinite path in the graph determines a semi-infinite geodesic in the group starting from e, and therefore a point on the boundary at infinity. In this way, we extend α_* to a map from infinite paths to $\partial\Gamma$.

A component of the automaton is a maximal subset in which any vertex can be reached from any other vertex. If there is a single non-trivial component, the recurrent part is transitive. This is the case for well-chosen automata for subgroups of $\mathrm{PSL}(2,\mathbb{R})$, but for general hyperbolic groups there is no such transitivity. Identifying points belonging to the same component, one obtains a new directed graph, the components graph, in which there is no loop. This graph encodes how different components interact.

We will denote by Σ^* the set of finite paths in the graph, by Σ the set of semi-infinite paths, and $\overline{\Sigma} = \Sigma^* \cup \Sigma$. These sets are endowed with a metric $d(\omega, \omega') = 2^{-n}$ where n is the first time the paths ω and ω' differ. With this metric, Σ^* is a dense open subset of the compact space $\overline{\Sigma}$. The map α_* is continuous from $\overline{\Sigma}$ to $\Gamma \cup \partial\Gamma$.

Denote by $\mathcal{H}^\beta$ the space of β-Hölder continuous functions on $\overline{\Sigma}$. For $0 < \beta < \beta'$, one has the following basic inequality (which is true in any metric space):

$$\|f\|_{\mathcal{H}^\beta} \leqslant 2 \|f\|_{C^0}^{1-\beta/\beta'} \|f\|_{\mathcal{H}^{\beta'}}^{\beta/\beta'}. \tag{3.1}$$

In particular, if a sequence of functions f_n converges in C^0 and remains bounded in $\mathcal{H}^{\beta'}$, then it converges in $\mathcal{H}^\beta$.

Note that an Hölder continuous function on Σ^* uniquely extends to an Hölder continuous function on $\overline{\Sigma}$. Finally, let $\sigma : \overline{\Sigma} \to \overline{\Sigma}$ be the left shift, forgetting the first edge of a path.

3.3. **Peripheral spectrum of transfer operators.** Since the spectral description of transfer operators is very classical, we will only sketch the proofs in this section, referring to [PP90] for more details.

Consider a finite directed graph $\mathcal{A}$, let $\overline{\Sigma}$ be the set of finite or infinite paths in $\mathcal{A}$, and let σ be the left shift. (If one is uncomfortable with the idea of considering finite paths in the graph, one can equivalently add a cemetery to the graph, that can be reached from any vertex, and extend a finite path by infinitely many steps in the cemetery.) To any real-valued Hölder continuous function $\varphi : \overline{\Sigma} \to \mathbb{R}$ (called a *potential*), one associates the so-called *transfer operator* $\mathcal{L}_\varphi$, defined on the set of Hölder continuous functions by

$$\mathcal{L}_\varphi f(\omega) = \sum_{\sigma(\omega')=\omega} e^{\varphi(\omega')} f(\omega'),$$

where for $\omega = \emptyset$ the empty path we only consider the non-empty preimages of ω. The iterates of this operator encode a lot of information on the Birkhoff sums $S_n\varphi(\omega) = \sum_{j=0}^{n-1} \varphi(\sigma^j \omega)$ of the potential φ. For instance, one has

$$\mathcal{L}_\varphi^n 1(\emptyset) = \sum e^{S_n\varphi(\omega)},$$

where the sum is over all paths of length n.

In the case of hyperbolic groups, we will be interested in the asymptotics of such sums, since for suitable potentials φ_r they correspond to the sum of H_r over the sphere of radius n in Γ (this is one of the quantity we want to estimate precisely to improve on Proposition 3.2). Such asymptotics can be read from the spectrum of $\mathcal{L}_\varphi$, that we now describe.

The simplest situation is when the graph is *topologically mixing*, i.e., one can go from any vertex to any other vertex (one says that the graph is recurrent) and for any $a, b \in \mathcal{A}$, for any large enough n, there is a path of length exactly n from a to b. In this case, the spectral description of $\mathcal{L}_\varphi$ is very simple, and is given by the following theorem (called the Ruelle-Perron-Frobenius theorem).

Theorem 3.6. *Assume $\mathcal{A}$ is topologically mixing. The operator $\mathcal{L}_\varphi$ acting on the space of Hölder continuous functions has a unique eigenvalue of maximal modulus denoted by $\mathrm{e}^{\Pr(\varphi)}$, the rest of its spectrum is contained in a disk of strictly smaller radius. Moreover, the corresponding eigenfunction h (suitably normalized) is strictly positive everywhere, and the eigenprojector is given by $\Pi f = \left(\int f \,\mathrm{d}\lambda\right) h$ for some probability measure λ whose support is the set Σ of infinite paths. Finally, the probability measure $h \,\mathrm{d}\lambda$ is invariant under σ and ergodic.*

In other words, one has

$$\left\| \mathcal{L}_\varphi^n f - e^{n \Pr(\varphi)} \left(\int f \,\mathrm{d}\lambda \right) h \right\| \leqslant C \left\| f \right\| e^{-n\varepsilon} e^{n \Pr(\varphi)},$$

for some $C > 0$ and $\varepsilon > 0$. This is Theorem 2.2 in [PP90] (the statement there is only given on Σ, but the proofs readily adapt to $\overline{\Sigma}$). The real number $\Pr(\varphi)$ is called the *pressure* of the potential φ.

Assume now that $\mathcal{A}$ is recurrent, but not mixing: there is a minimal period $p > 1$ such that any path from a vertex to itself has length np for some integer n. In this case, the set V of vertices of $\mathcal{A}$ is a disjoint union $\bigsqcup_{j=0}^{p-1} V_j$, where for any $j \in \mathbb{Z}/p\mathbb{Z}$ an outgoing edge of V_j is an ingoing edge of V_{j+1} (we call this decomposition a cyclic decomposition of V). Denoting by $\overline{\Sigma}_j$ the set of paths beginning from a vertex in V_j and the empty path, then σ maps $\overline{\Sigma}_j$ to $\overline{\Sigma}_{j+1}$. Moreover, the restriction of σ^p to any $\overline{\Sigma}_j$ is a topologically mixing subshift of finite type, to which Theorem 3.6 applies. This readily implies that the eigenvalues of maximal modulus of $\mathcal{L}_\varphi$ are of the form $e^{2\mathrm{i}k\pi/p}e^{\Pr(\varphi)}$ for some real number $\Pr(\varphi)$, they are all simple, and the rest of the spectrum of $\mathcal{L}_\varphi$ is contained in a disk of strictly smaller radius. More specifically, there exist positive functions h_j on $\overline{\Sigma}_j$ and probability measures λ_j with support equal to Σ_j such that

$$\left\| \mathcal{L}_\varphi^n f - e^{n\Pr(\varphi)} \sum_{j=0}^{p-1} \left(\int f \,\mathrm{d}\lambda_{(j-n \bmod p)} \right) h_j \right\| \leqslant C \left\| f \right\| e^{-n\varepsilon} e^{n\Pr(\varphi)}.$$

Assume finally that $\mathcal{A}$ is not even recurrent. In this case, one can associate to any component $\mathcal{C}$ the restriction of φ to paths staying in $\mathcal{C}$ and the corresponding transfer operator $\mathcal{L}_\mathcal{C}$. The previous description applies to $\mathcal{L}_\mathcal{C}$: it has finitely many eigenvalues of maximal modulus $e^{\Pr_\mathcal{C}(\varphi)}$, they are of the form $e^{2\mathrm{i}k\pi/p_\mathcal{C}}e^{\Pr_\mathcal{C}(\varphi)}$ for some $k \in \mathbb{Z}/p_\mathcal{C}\mathbb{Z}$, and $\mathcal{L}_\mathcal{C}$ has a spectral gap. Let $\Pr(\varphi)$ be the maximum of $\Pr_\mathcal{C}(\varphi)$ over all components. We call a component maximal if $\Pr_\mathcal{C}(\varphi) = \Pr(\varphi)$. The dominating terms in $\mathcal{L}_\varphi^n$ come from the maximal components. We will say that φ is *semisimple* if there is no directed path from a maximal component to a different maximal component. Otherwise, the eigenvalue $e^{\Pr(\varphi)}$ has non-trivial Jordan blocks, which makes the precise spectral description more cumbersome.

Lemma 3.7. *Consider some edge e_0, and let $k > 0$ be such that there is a path from e_0 to successively k different maximal components. For any nonnegative function f with $f \geqslant 1$ on the set of paths starting with e_0, one has $\mathcal{L}_\varphi^n f(\emptyset) \geqslant Cn^{k-1}e^{n\Pr(\varphi)}$.*

In the semisimple case, the asymptotics of $\mathcal{L}_\varphi^n$ can be described as follows.

Theorem 3.8. *Assume that φ is semisimple. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_I$ the maximal components, with corresponding period p_i, and consider for each i a cyclic decomposition $\mathcal{C}_i = \bigsqcup_{j \in \mathbb{Z}/p_i\mathbb{Z}} \mathcal{C}_{i,j}$. There exist functions $h_{i,j}$ and measures $\lambda_{i,j}$ with $\int h_{i,j}\,\mathrm{d}\lambda_{i,j} = 1$ such that*

$$\left\| \mathcal{L}_\varphi^n f - e^{n\Pr(\varphi)} \sum_{i=1}^{I} \sum_{j=0}^{p_i-1} \left(\int f \,\mathrm{d}\lambda_{i,(j-n \bmod p_i)} \right) h_{i,j} \right\| \leqslant C \left\| f \right\| e^{-n\varepsilon} e^{n\Pr(\varphi)}.$$

The probability measures $\mathrm{d}\mu_i = \frac{1}{p_i}\sum_{j=0}^{p_i-1} h_{i,j}\,\mathrm{d}\lambda_{i,j}$ are invariant under σ and ergodic.

Denote by $\mathcal{C}_{\to,i,j}$ the set of edges from which one can reach $\mathcal{C}_{i,j}$ with a path of length in $p_i\mathbb{N}$, and by $\mathcal{C}_{i,j,\to}$ the set of edges that can be reached from $\mathcal{C}_{i,j}$ by a path of length in $p_i\mathbb{N}$. The function $h_{i,j}$ is bounded from below on paths beginning by an edge in $\mathcal{C}_{i,j,\to}$ (and the *empty path) and vanishes elsewhere. The support of the measure $\lambda_{i,j}$ is the set of infinite paths beginning in $\mathcal{C}_{\to,i,j}$ with infinitely many coordinates in $\mathcal{C}_i$.*

Proof. The lemma and the theorem are basic linear algebra once Theorem 3.6 is given. Indeed, one can decompose $\mathcal{L}$ into a sum of operators corresponding to edges in the components graph. Since there is no loop in this graph, this is a Jordan-blocks like decomposition, which readily gives the dominating spectrum of $\mathcal{L}$ from the dominating spectrum on each component. The only nontrivial assertion is on the support of $h_{i,j}$ and $\lambda_{i,j}$ in the theorem. When there is only one non-trivial component and this component is mixing, the argument is given in [GL11, Theorem 6.1]. It easily extends to the general case. □

We will need the following simple lemma later on:

Lemma 3.9. *Under the assumptions of the above theorem, let $\beta_i = \sum_j \lambda_{i,j}$. Then $\sigma_*\beta_i$ is absolutely continuous with respect to β_i.*

Proof. The measures $\lambda_{i,j}$ are constructed as eigenmeasures of the operator $(\mathcal{L}_\varphi^*)^{p_i}$. More precisely, they satisfy $\mathcal{L}_\varphi^*\lambda_{i,j} = e^{\Pr(\varphi)}\lambda_{i,(j-1 \bmod p_i)}$. In particular, $\mathcal{L}_\varphi^*\beta_i = e^{\Pr(\varphi)}\beta_i$.

Consider a cylinder $[\omega_0, \dots, \omega_n]$, i.e., the set of paths that start with those symbols. The function $\mathcal{L}_\varphi 1_{[\omega_0,\dots,\omega_n]}$ is uniformly bounded on the image of this cylinder under σ, i.e., $[\omega_1, \dots, \omega_n]$, and it vanishes elsewhere. Hence,

$$\beta_i([\omega_0, \dots, \omega_n]) = e^{-\Pr(\varphi)}\mathcal{L}_\varphi^*\beta_i(1_{[\omega_0,\dots,\omega_n]} = e^{-\Pr(\varphi)}\beta_i(\mathcal{L}_\varphi 1_{[\omega_0,\dots,\omega_n]}) \leqslant C\beta_i([\omega_1, \dots, \omega_n]).$$

Since $\sigma^{-1}([\omega_1, \dots, \omega_n])$ is a finite union of cylinders of the form $[\omega_0, \dots, \omega_n]$, we obtain $\beta_i(\sigma^{-1}[\omega_1, \dots, \omega_n]) \leqslant C\beta_i([\omega_1, \dots, \omega_n])$. As cylinders generate the topology, it follows that $\beta_i(\sigma^{-1}A) \leqslant C\beta_i(A)$ for any measurable set A. □

Finally, we will need to describe what happens under perturbations of the potential.

Proposition 3.10. *Let $\varphi \in \mathcal{H}^\beta$ be a semisimple Hölder potential, with maximal components $\mathcal{C}_1, \dots, \mathcal{C}_I$ and spectral description as in Theorem 3.8. There exist $\varepsilon > 0$ and $C > 0$ such that, for any ψ which is small enough in $\mathcal{H}^\beta$, there exist functions $h_{i,j}^\psi$ and measures $\lambda_{i,j}^\psi$ (with the same support as, respectively, $h_{i,j}$ and $\lambda_{i,j}$) and numbers $\Pr_i(\varphi+\psi)$ with*

$$\left\| \mathcal{L}_{\varphi+\psi}^n f - \sum_{i=1}^{I} e^{n\Pr_i(\varphi+\psi)} \sum_{j=0}^{p_i-1} \left(\int f \,\mathrm{d}\lambda_{i,(j-n \bmod p_i)}^\psi \right) h_{i,j}^\psi \right\| \leqslant C \|f\| e^{-n\varepsilon} e^{n\Pr(\varphi)}.$$

The maps $\psi \mapsto \Pr_i(\varphi+\psi)$, $\psi \mapsto h_{i,j}^\psi$ and $\psi \mapsto \lambda_{i,j}^\psi$ are real analytic from a small ball around 0 in $\mathcal{H}^\beta$ to, respectively, $\mathbb{R}$, $\mathcal{H}^\beta$ and the dual of $\mathcal{H}^\beta$. Finally,

$$\Pr_i(\varphi+\psi) = \Pr(\varphi) + \int \psi \,\mathrm{d}\mu_i + O(\|\psi\|_{\mathcal{H}^\beta}^2),$$

where $\mathrm{d}\mu_i = \frac{1}{p_i}\sum_{j=0}^{p_i-1} h_{i,j}\,\mathrm{d}\lambda_{i,j}$.

Proof. Let us first assume that the shift is topologically mixing. In this case, the dominating eigenvalue $e^{\Pr(\varphi)}$ of $\mathcal{L}_\varphi$ is simple. Simple isolated eigenvalues and the corresponding eigenprojectors and eigenfunctions depend in an analytic way on the operator, by classical perturbation theory [Kat66]. Moreover, by semicontinuity of the spectrum, the perturbed operators $\mathcal{L}_{\varphi+\psi}$ also have a spectral gap, uniformly in ψ close enough to 0. One gets

$$\mathcal{L}^n_{\varphi+\psi} f = e^{n\,\mathrm{Pr}(\varphi+\psi)} \left(\int f \,\mathrm{d}\lambda^\psi \right) h^\psi + O(e^{-n\varepsilon} e^{n\,\mathrm{Pr}(\varphi)})$$

for some $\mathrm{Pr}(\varphi+\psi)$, λ^ψ and h^ψ that depend analytically on ψ. This almost completes the proof of the theorem in this case, it only remains to show that $\mathrm{Pr}(\varphi+\psi) = \mathrm{Pr}(\varphi) + \int \psi h \,\mathrm{d}\lambda + O(\|\psi\|^2)$. By analyticity, it is sufficient to show that the derivative of the pressure at 0 is given by the integral with respect to the measure $h\,\mathrm{d}\lambda$. This is [PP90, Proposition 4.10].

The topologically transitive case readily reduces to the mixing case by considering σ^p where p is the period.

In the general case, one obtains different pressures $\mathrm{Pr}_i(\varphi+\psi)$ on each component $\mathcal{C}_i$. On other components that were not maximal for φ, the pressure of $\varphi+\psi$ remains bounded away from $\mathrm{Pr}(\varphi)$. It follows that the maximal components of $\varphi+\psi$ are contained in those of φ if ψ is small enough. In particular, $\varphi+\psi$ is semisimple, and $\mathrm{Pr}(\varphi+\psi) = \max \mathrm{Pr}_i(\varphi+\psi)$. Finally, the spectral description of $\mathcal{L}_{\varphi+\psi}$ follows from the description on each component $\mathcal{C}_i$ separately. □

3.4. **Transfer operators in hyperbolic groups.** Let Γ be a non-elementary Gromov-hyperbolic group, and μ a probability measure satisfying strong Ancona inequalities. Consider a strongly Markov automatic structure for Γ, given by a directed graph $\mathcal{A} = (V, E, s_*)$ and a labeling $\alpha : E \to S$ of edges by generators of the group. We will use freely the notations of Paragraph 3.2.

For $r \in [1, R]$, let us define a potential φ_r on the set Σ^* of finite paths in the automaton by

$$\varphi_r(\omega) = \log \left(\frac{H_r(e, \alpha_*(\omega))}{H_r(e, \alpha_*(\sigma\omega))} \right).$$

Consider a path $\omega = \omega_0 \cdots \omega_{n-1}$ of length n, then

$$e^{S_n \varphi_r(\omega)} = \frac{H_r(e, \alpha_*(\omega_0 \cdots \omega_{n-1}))}{H_r(e, e)}.$$

Let E_* be the set of edges starting from the vertex s_* of the graph $\mathcal{A}$, and let $1_{[E_*]}$ be the function equal to 1 on paths starting with an edge in E_*, and 0 elsewhere. Using the language of transfer operators, we have

$$\begin{aligned} H_r(e,e) \cdot \mathcal{L}^n_{\varphi_r} 1_{[E_*]}(\emptyset) &= H_r(e,e) \sum_{\omega = \omega_0 \cdots \omega_{n-1}} e^{S_n \varphi_r(\omega)} 1(\omega_0 \in E_*) \\ &= \sum H_r(e, \alpha_*(\omega_0 \cdots \omega_{n-1})) 1(\omega_0 \in E_*). \end{aligned}$$

Since α_* induces a bijection between the paths of length n starting from s_* and the sphere $\mathbb{S}_n$ of radius n in Γ, we obtain

$$\sum_{x \in \mathbb{S}_n} H_r(e, x) = H_r(e,e) \mathcal{L}^n_{\varphi_r} 1_{[E_*]}(\emptyset).$$

Therefore, the spectrum of $\mathcal{L}_{\varphi_r}$ will give asymptotics for $\sum_{x \in \mathbb{S}_n} H_r(e, x)$. To be able to use the results of the previous paragraph, one should check that φ_r is Hölder continuous.

Lemma 3.11. *There exists $\beta > 0$ such that, for any $r \in [1, R]$, the function φ_r is Hölder continuous of exponent β on Σ^*. Therefore, it extends to an Hölder continuous function on $\overline{\Sigma}$, that we still denote by φ_r. It satisfies $\|\varphi_r\|_{\mathcal{H}^\beta} \leqslant C$, uniformly in $r \in [1, R]$. Moreover,*

$$\|\varphi_r - \varphi_R\|_{\mathcal{H}^\beta} \leqslant C(R-r)^{1/3}. \tag{3.2}$$

Proof. Consider two finite paths ω and ω' with $d(\omega, \omega') = 2^{-n} < 1$, so that they match up to length $n \geqslant 1$. In particular, $\omega_0 = \omega'_0$. Let $x = \alpha_*(\omega)$, $x' = \alpha_*(\omega')$ and $a = \alpha_*(\omega_0)$, so that $\varphi_r(\omega) = \log(H_r(e,x)/H_r(a,x))$ and $\varphi_r(\omega') = \log(H_r(e,x')/H_r(a,x'))$. The points e, a and x, x' are in the situation of strong Ancona inequalities (Definition 2.8) with a separating distance $n-1$. Since μ satisfies strong uniform Ancona inequalities, it follows that $|\varphi_r(\omega) - \varphi_r(\omega')| \leqslant Ce^{-\rho n}$ for some $\rho > 0$. Hence, for some $\beta' > 0$, φ_r belongs to $\mathcal{H}^{\beta'}$ and is uniformly bounded in this space.

Lemma 3.4 implies that $\|\varphi_r - \varphi_R\|_{C^0} \leqslant C(R-r)^{1/2}$. Together with the uniform boundedness of φ_r in $\mathcal{H}^{\beta'}$, this shows that $\|\varphi_r - \varphi_R\|_{\mathcal{H}^\beta} \leqslant C(R-r)^{1/3}$ if β is small enough, by (3.1).

Finally, we have proved all those inequalities on the space Σ^* of finite paths. Since Hölder continuous functions on Σ^* extend to Hölder continuous functions on $\overline{\Sigma}$, the result follows. □

Remark 3.12. One could in fact show that $\|\varphi_r - \varphi_R\|_{\mathcal{H}^\beta} \leqslant C(R-r)^{1/2}$ by mimicking the proof of Lemma 3.4 at the level of Hölder exponents. Since this computation is lengthy and (3.2) will be sufficient for our purposes, we omit it.

Lemma 3.13. *We have* $\Pr(\varphi_R) = 0$. *Moreover, φ_R is semisimple.*

Proof. Suppose $\Pr(\varphi_R) < 0$. Then $\mathcal{L}^n_{\varphi_R} 1_{[E_*]}$ goes to zero exponentially fast in the space of Hölder functions. In particular, $\sum_{x \in \mathbb{S}_n} H_R(e,x) = H_R(e,e)\mathcal{L}^n_{\varphi_R} 1_{[E_*]}(\emptyset)$ is exponentially small. One can use this estimate to prove that the series $G_{R+\varepsilon}(e,e)$ converges for some $\varepsilon > 0$: this is the content of the proof of Proposition 7.1 in [GL11] (the proof is written for symmetric measures, but it applies equally well in non-symmetric situations). This is a contradiction since, by definition, R is the radius of convergence of the series $G_r(e,e)$. Hence, $\Pr(\varphi_R) \geqslant 0$.

If $\Pr(\varphi_R)$ were strictly positive, or $\Pr(\varphi_R) = 0$ but φ_R were not semisimple, then Lemma 3.7 would imply that $\mathcal{L}^n_{\varphi_R} 1_{[E_*]}(\emptyset)$ would tend to infinity. This quantity is equal to $H_R(e,e)^{-1} \sum_{x \in \mathbb{S}_n} H_R(e,x)$. Since it remains bounded by Lemma 2.5, we obtain a contradiction. □

One can now come back to Corollary 3.3. Since $\Pr(\varphi_R) = 0$ and φ_R is semisimple, Theorem 3.8 implies in particular that $\mathcal{L}^n_{\varphi_R} 1_{[E_*]}(\emptyset)$ is bounded from below. Since it coincides with $H_R(e,e)^{-1} \sum_{x \in \mathbb{S}_n} H_R(e,x)$, we get $\sum_{x \in \Gamma} H_R(e,x) = +\infty$. This shows that the constant A in Corollary 3.3 vanishes, and therefore

$$\frac{C^{-1}}{\sqrt{R-r}} \leqslant \sum_{x \in \Gamma} H_r(e,x) \leqslant \frac{C}{\sqrt{R-r}}. \tag{3.3}$$

Let us introduce a convenient notation: we will reserve the notation $\tau(r)$ (possibly with some indices) for continuous functions of r taking values in $(0, +\infty)$ that extend continuously up to $r = R$ and are bounded away from zero.

We will now use the spectral perturbation given by Proposition 3.10 to study $\mathcal{L}_{\varphi_r}$. If r is close to R, then $\varphi_r - \varphi_R$ is small in $\mathcal{H}^\beta$ by Lemma 3.11. Applying the proposition on spectral perturbation to the function $f = 1_{[E_*]}$, we get the following. Let p be the least common multiple of the periods of the maximal components of φ_R. For any $q \in [0, p)$, one has (since $\Pr(\varphi_R) = 0$)

$$\mathcal{L}_{\varphi_r}^{np+q} 1_{[E_*]}(\emptyset) = \sum_{i=1}^{I} e^{(np+q)\Pr_i(\varphi_r)} \tau_0(q, i, r) + O(e^{-n\varepsilon}),$$

for some functions $\tau_0(q,i,r)$ (as in the notation we introduced in the previous paragraph). Since this is equal to $H_r(e,e)^{-1} \sum_{x \in \mathbb{S}_{np+q}} H_r(e,x)$ and since $\sum_{x\in\Gamma} H_r(e,x) < \infty$, it follows in particular that $\Pr_i(\varphi_r)$ is strictly negative for all i.

Summing over n and q, we get

$$\sum_{x\in\Gamma} H_r(e,x) = H_r(e,e) \sum_{n,q} \mathcal{L}_{\varphi_r}^{np+q} 1_{[E_*]}(\emptyset) = H_r(e,e) \sum_{q=0}^{p-1} \sum_{i=1}^{I} \frac{e^{q \Pr_i(\varphi_r)}}{1 - e^{p \Pr_i(\varphi_r)}} \tau_0(q,i,r) + O(1)$$

$$= \sum_{i=1}^{I} \frac{\tau_1(i,r)}{|\Pr_i(\varphi_r)|} + O(1), \tag{3.4}$$

for some functions $\tau_1(i,r)$.

By (3.3), $|\Pr(\varphi_r)| = \inf_i |\Pr_i(\varphi_r)|$ is comparable to $\sqrt{R-r}$. It will be important to show that all the $|\Pr_i(\varphi_r)|$ are of the same order of magnitude: otherwise, some components would not play a significant role for $r < R$ while they would become important at $r = R$, ruining the continuity properties we are seeking. This is the main difference with the transitive situation, where there is only one eigenvalue to consider.

Theorem 3.14. *For any $i \in [1, I]$, the ratio $\Pr_i(\varphi_r)/\Pr(\varphi_r)$ tends to 1 when $r \to R$.*

We will prove this theorem in the next subsection. It follows from this result that

$$\sum_{x\in\Gamma} H_r(e,x) = \frac{\tau_2(r)}{|\Pr(\varphi_r)|} + O(1). \tag{3.5}$$

Hence, the spectral data of $\mathcal{L}_{\varphi_r}$ are directly related to the function $\eta(r) = \sum_{x\in\Gamma} H_r(e,x)$.

3.5. **Pressure does not depend on the component.** In this subsection, we prove Theorem 3.14. We will in particular rely on the estimate $\|\varphi_r - \varphi_R\|_{\mathcal{H}^\beta} \leqslant C(R-r)^{1/3}$ from (3.2), that in turn was proved using the a priori estimates from Lemma 3.4.

By Proposition 3.10, the variation of the pressure mainly depends on the integral $\int (\varphi_r - \varphi_R) \,\mathrm{d}\mu_i$. We will show that this integral does not depend on i, using a geometric argument in the group due to [CF10].

Fix some $r \in [1, R]$. For $c \in \mathbb{R}$, we define a set $U(c) \subset \partial\Gamma$ as the set of points ξ such that, along some geodesic from e to ξ, $\log H_r(e,x)/d(e,x) \to c$. Equivalently, this convergence holds along any geodesic tending to ξ, and one can replace $H_r(e,x)$ with $H_r(a,x)$ and $d(e,x)$ with $d(b,x)$ for any $a, b \in \Gamma$. Indeed, geodesics tending to ξ remain within a bounded distance from each other, by [GdlH90, Proposition 7.2] (therefore, by Harnack inequalities H_r varies by at most a multiplicative constant when one changes geodesics), and the ratio $H_r(e,x)/H_r(a,x)$ also remains bounded from above and from below again by Harnack inequalities. In particular, $U(c)$ is invariant under the action of Γ: for any $g \in \Gamma$, $g \cdot U(c) = U(c)$.

Let $c_i = \int \varphi_r \,\mathrm{d}\mu_i$, we will show that, for all $i \neq i'$, $g \cdot U(c_i)$ intersects $U(c_{i'})$ for some $g \in \Gamma$. This will give $U(c_i) = U(c_{i'})$, hence $c_i = c_{i'}$ as desired. To prove this, we will show that the sets $U(c_i)$ all have positive measure for some measure on $\partial\Gamma$ which is ergodic under the action of Γ.

Let us first construct the measure. Let p be the least common multiple of the periods p_i, and fix $q \in [0,p)$. It follows from the spectral description of $\mathcal{L}_{\varphi_R}$ (Theorem 3.8) that, for any Hölder continuous function f on $\overline{\Sigma}$, $\mathcal{L}_{\varphi_R}^{np+q} f(\emptyset)$ converges when $n \to \infty$. In turn, this convergence follows for any continuous function, by approximation (since the iterates of $\mathcal{L}_{\varphi_R}$ on C^0 remain bounded, since $\mathcal{L}_{\varphi_R}^n 1$ itself remains bounded). If f is a continuous function on $\Gamma \cup \partial\Gamma$, then $\sum_{x \in \mathbb{S}_n} H_R(e,x) f(x) = H_R(e,e) \mathcal{L}_{\varphi_R}^n (1_{[E_*]} \cdot f \circ \alpha_*)(\emptyset)$, and $f \circ \alpha_*$ is continuous. Let us define a measure m_n supported on $\mathbb{S}_n$ by $m_n = \sum_{x \in \mathbb{S}_n} H_R(e,x)\delta_x$, this shows that the sequence of measures m_{np+q} converges to a limiting measure (which is supported on $\partial\Gamma$, and has mass bounded from above and from below). We deduce that the measures $(\sum_1^N m_n)/(\sum_1^N m_n(\Gamma))$ converge to a probability measure on $\partial\Gamma$, that we denote by ν_R. It also follows that this measure can be constructed using the Patterson-Sullivan technique: the measures

$$\theta_s = \sum_{x \in \Gamma} H_R(e,x) e^{-s|x|} \delta_x \Big/ \sum_{x \in \Gamma} H_R(e,x) e^{-s|x|} \tag{3.6}$$

are well defined for $s > 0$, and they converge when s tends to 0 towards ν_R.

For $g \in \Gamma$, let us denote by L_g the left multiplication by Γ. Then, for any $x \in \Gamma$,

$$(L_g)_* \theta_s(x) = \theta_s(g^{-1}x) = \frac{H_R(e, g^{-1}x)}{H_R(e,x)} e^{-s(|g^{-1}x| - |x|)} \theta_s(x) = \tilde{K}_x(g) e^{-s(|g^{-1}x| - |x|)} \theta_s(x),$$

where $\tilde{K}_x(g) = H_R(g,x)/H_R(e,x)$ is the Martin Kernel associated to H_R. When x tends to a point $\xi \in \partial\Gamma$, this quantity converges to a limit denoted by $\tilde{K}_\xi(g)$. Since $|g^{-1}x| - |x|$ is uniformly bounded when x varies in Γ, we deduce letting s tend to 0 that

$$\frac{\mathrm{d}(L_g)_* \nu_R}{\mathrm{d}\nu_R}(\xi) = \tilde{K}_\xi(g). \tag{3.7}$$

Following the classical arguments of Patterson-Sullivan (due in this context to [Coo93] and [BHM11]), we deduce the following:

Proposition 3.15. *The measure ν_R is ergodic for the action of Γ.*

Proof. We want to apply the results of [Coo93] and [BHM11] saying that a Patterson-Sullivan measure is ergodic. Thus, we should interpret the function $\tilde{K}_\xi(g)$ in (3.7) as the exponential of a Busemann cocycle. Since $\tilde{K}_\xi(g)$ is the limit of $H_R(g,x)/H_R(e,x)$, the function $\log \tilde{K}_\xi$ would be the Busemann cocycle associated to a distance $\tilde{d}$ if $H_R(x,y) = Ce^{-\tilde{d}(x,y)}$, for some constant C. Let us therefore set $\tilde{d}(x,y) = -\log(F_R(x,y)F_R(y,x))$, where F_R is the first visit Green function. We should show that $\tilde{d}$ is a distance, that it is equivalent to d, and hyperbolic, to be able to apply the results of [Coo93] and [BHM11].

The subadditivity (2.3) of F_R shows that $\tilde{d}$ satisfies the triangular inequality. For $x \neq y$, considering n concatenations of paths from x to y then to x, one gets

$$G_R(x,x) \geqslant \sum_{n=0}^{\infty} (F_R(x,y)F_R(y,x))^n.$$

Since $G_R(x,x)$ is finite, this shows that $F_R(x,y)F_R(y,x) < 1$. Hence, $\tilde{d}$ is a distance. It is a variant of the Green distance studied in [BHM11].

The quantity $G_R(e,e)$, which is finite, equals $\sum_\gamma w_R(\gamma)$ (where the sum is over all paths from e to itself, and the notation $w_R(\gamma)$ for the R-weight of a path γ has been introduced in Subsection 2.1). Excluding finitely many paths, one can make the remaining sum arbitrarily small. If x is not on one of those finitely many paths, then $F_R(e,x)F_R(x,e)$ is bounded by the remaining sum, and is therefore arbitrarily small. This shows that $\tilde{d}(e,x)$ tends to infinity when $x \to \infty$ in Γ.

Since G_R (or, equivalently, F_R) satisfies Ancona inequalities, there exists $D > 0$ such that $\tilde{d}(x,z) \geqslant \tilde{d}(x,y) + \tilde{d}(y,z) - D$ whenever x, y, z are on a geodesic segment in this order. Let L be such that $\tilde{d}(e,x) \geqslant 2D$ for $|x| \geqslant L$. By induction, this implies that $\tilde{d}(e,x) \geqslant (n+1)D$ for $|x| \geqslant nL$. In particular, there exists a constant $C > 0$ such that, for all x, $\tilde{d}(e,x) \geqslant C^{-1}|x|$. By Harnack inequalities (2.2), we also have $\tilde{d}(e,x) \leqslant C|x|$. This shows that the distance $\tilde{d}$ is equivalent to the word distance d.

The word distance is hyperbolic. It does not immediately follow that $\tilde{d}$ is hyperbolic, since the metric space $(\Gamma, \tilde{d})$ is usually not geodesic (while the invariance of hyperbolicity under quasi-isometries requires such an assumption). However, [BHM11] proves that if Ancona inequalities hold then $\tilde{d}$ is hyperbolic (the proof given in their Theorem 1.1 is for the usual Green metric, but it applies verbatim in our setting).

Finally, we can apply the results of Paragraphs 2.2 and 2.3 in [BHM11]. The equation (3.7) shows that ν_R is quasi-conformal for a distance at infinity coming from the hyperbolic distance $\tilde{d}$ on Γ. Therefore, [BHM11, Theorem 2.7] implies that ν_R is ergodic. □

Proposition 3.16. *For $i \neq i'$, one has $\int \varphi_r \,\mathrm{d}\mu_i = \int \varphi_r \,\mathrm{d}\mu_{i'}$.*

Proof. The limit of $\mathcal{L}_{\varphi_R}^{np+q} f(\emptyset)$ is given by $\sum_{i=1}^{I} \sum_{j=0}^{p_i-1} \left(\int f \,\mathrm{d}\lambda_{i,(j-q \bmod p_i)}\right) h_{i,j}(\emptyset)$. Since $h_{i,j}(\emptyset)$ is bounded from above and from below, we deduce that ν_R is equivalent to the push-forward under α_* of the measure $\sum_{i,j} \lambda_{i,j}$ restricted to the set of paths beginning with an edge in E_*.

The probability measure $\mathrm{d}\mu_i = \frac{1}{p_i} \sum_{j=0}^{p_i-1} h_{i,j} \,\mathrm{d}\lambda_{i,j}$ is invariant and ergodic. Let $O_i \subset \Sigma$ denote the set of points such that the normalized Birkhoff sums $S_n f/n$ converge to $\int f \,\mathrm{d}\mu_i$ for any continuous function f. By Birkhoff ergodic theorem, $\mu_i(O_i) = 1$. Since μ_i is equivalent to $\beta_i = \sum_j \lambda_{i,j}$ restricted to the set $\overline{\Sigma}_i$ of paths staying in the component $\mathcal{C}_i$, we get $\beta_i(O_i^c \cap \overline{\Sigma}_i) = 0$ (where O_i^c denotes the complement of O_i). We deduce that

$$\beta_i(O_i^c) = 0. \tag{3.8}$$

Otherwise, since β_i-almost every point ends up in $\overline{\Sigma}_i$ after finitely many iterations, we would have $\beta_i(O_i^c \cap \sigma^{-k}\overline{\Sigma}_i) > 0$ for some $k \geqslant 0$, hence $\beta_i(\sigma^{-k}(\sigma^k O_i^c \cap \overline{\Sigma}_i)) > 0$. Since $\sigma_*^k \beta_i$ is absolutely continuous with respect to β_i by Lemma 3.9, and $\sigma^k O_i^c \subset O_i^c$, this gives $\beta_i(O_i^c \cap \overline{\Sigma}_i) > 0$, a contradiction.

Let us now show that

$$\nu_R(\alpha_*(O_i \cap [E_*])) > 0, \tag{3.9}$$

where $O_i \cap [E_*]$ denotes the set of paths in O_i beginning with an edge in E_*. Otherwise, since the image of $\beta_i(\cdot \cap [E_*])$ is absolutely continuous with respect to ν_R, we would get $(\alpha_*\beta_i)(\alpha_*(O_i \cap [E_*])) = 0$, hence $\beta_i(O_i \cap [E_*]) = 0$. Since $\beta_i(O_i^c) = 0$ by (3.8), we get $\beta_i([E_*]) = 0$. This is a contradiction since Theorem 3.8 shows that β_i gives positive weight to $[E_*]$.

Consider now $\omega \in O_i \cap [E_*]$, and let $\xi = \alpha_*(\omega) \in \partial\Gamma$. The path $\alpha(\omega)$ is a geodesic converging to ξ. In particular, denoting by $\bar\omega_n$ the beginning of ω of length n, $x_n = \alpha_*(\bar\omega_n)$ is a sequence of points converging to ξ along a geodesic ray. Moreover,

$$\log H_r(e, x_n) = S_n\varphi_r(\bar\omega_n) + \log H_r(e, e).$$

Since φ_r is Hölder continuous, $S_n\varphi_r(\bar\omega_n) - S_n\varphi_r(\omega)$ remains uniformly bounded. Hence, $\log H_r(e, x_n)/n = S_n\varphi_r(\omega)/n + o(1)$ tends to $c_i = \int \varphi_r \,\mathrm{d}\mu_i$ by definition of O_i. This shows that $\xi \in U(c_i)$. Therefore, $\alpha_*(O_i \cap [E_*]) \subset U(c_i)$. With (3.9), this gives $\nu_R(U(c_i)) > 0$.

Since ν_R is ergodic for the action of Γ by Proposition 3.15, and the sets $U(c)$ are Γ-invariant, we deduce that $U(c_i)$ has full measure. Therefore, all those sets have to coincide. □

Proof of Theorem 3.14. By Proposition 3.10, the pressure $\mathrm{Pr}_i(\varphi_r)$ on the component $\mathcal{C}_i$ is equal to $\int(\varphi_r - \varphi_R)\,\mathrm{d}\mu_i + O(\|\varphi_r - \varphi_R\|)^2$. The integral does not depend on i, by Proposition 3.16. Considering i' such that the pressure is maximal, we obtain

$$\mathrm{Pr}_i(\varphi_r) = \mathrm{Pr}(\varphi_r) + O(\|\varphi_r - \varphi_R\|)^2.$$

Moreover, by (3.3) and (3.4), the ratio between $\mathrm{Pr}(\varphi_r)$ and $-\sqrt{R-r}$ is bounded from above and below. Since $\|\varphi_r - \varphi_R\|^2 = O(R-r)^{2/3}$ by Lemma 3.11, we obtain $\mathrm{Pr}_i(\varphi_r) = \mathrm{Pr}(\varphi_r) + o(\mathrm{Pr}(\varphi_r))$. This concludes the proof. □

3.6. **Estimating the second derivative of the Green function.** To improve on Proposition 3.2, one should get asymptotics for the function $\sum_{x,y} G_r(e,y)G_r(y,x)G_r(x,e)$ in terms of $\eta(r) = \sum G_r(e,x)G_r(x,e) = \sum H_r(e,x)$ or, equivalently, in terms of $\mathrm{Pr}(\varphi_r)$.

Proposition 3.17. *One has when $r \to R$*

$$\sum_{x,y} G_r(e,y)G_r(y,x)G_r(x,e) = c(r)\eta(r)^3 + O(\eta(r)^2), \tag{3.10}$$

for some nonnegative function $c(r)$ that extends continuously to $r = R$.

This subsection is devoted to the proof of this proposition. We will need to express things in terms of transfer operators on the symbolic space. Let

$$\Phi_r(x) = \frac{\sum_y G_r(e,y)G_r(y,x)}{G_r(e,x)}, \tag{3.11}$$

and define for $r < R$ a probability measure ν_r on Γ by

$$\int f \,\mathrm{d}\nu_r = \frac{\sum_{x\in\Gamma} H_r(e,x)f(x)}{\sum_{x\in\Gamma} H_r(e,x)}. \tag{3.12}$$

The sum in (3.10) is equal to

$$\eta(r) \int \Phi_r \, \mathrm{d}\nu_r.$$

To estimate it, we should understand Φ_r and ν_r.

Proposition 3.18. *When $r \to R$, the sequence of probability measures ν_r on the compact space $\Gamma \cup \partial\Gamma$ converges weakly to a probability measure ν_R, which is supported on $\partial\Gamma$.*

Proof. If ν_r converges weakly, then the limiting measure can give no weight to Γ, since $\nu_r(x) = H_r(e,x)/\sum_{y\in\Gamma} H_r(e,y)$ tends to 0 by (3.3).

Therefore, we just have to prove the convergence of $\nu_r(f)$ for any continuous function, or even for f in a dense set of functions. We will consider those f such that the function $\tilde{f}$ defined on Σ^* by $\tilde{f}(\omega) = f(\alpha_*(\omega))$ belongs to $\mathcal{H}^\beta$. For such a function, we have

$$\sum_{x\in\Gamma} H_r(e,x)f(x) = H_r(e,e) \sum_{n\in\mathbb{N}} \mathcal{L}^n_{\varphi_r}(1_{[E_*]}\tilde{f})(\emptyset).$$

Using the spectral description of Proposition 3.10, we deduce that this can be written as

$$\sum_{i=1}^{I} c_{\tilde{f}}(i,r)/\left|\mathrm{Pr}_i(\varphi_r)\right| + O(1),$$

as in (3.4), for some functions $c_{\tilde{f}}(i,r)$ that extend continuously up to $r = R$. Therefore, by (3.5),

$$\nu_r(f) = \frac{\sum_{i=1}^{I} c_{\tilde{f}}(i,r)/\left|\mathrm{Pr}_i(\varphi_r)\right| + O(1)}{\tau_2(r)/\left|\mathrm{Pr}(\varphi_r)\right| + O(1)}.$$

Since all the quantities $|\mathrm{Pr}_i(\varphi_r)|$ are asymptotic to $|\mathrm{Pr}(\varphi_r)|$ by Theorem 3.14 and tend to 0, this converges when r tends to R (to $\sum_{i=1}^{I} c_{\tilde{f}}(i,R)/\tau_2(R)$). □

Remark 3.19. One can easily check that the measure ν_R in Proposition 3.18 is the same as the measure we constructed in Subsection 3.5 and was already denoted by ν_R. This will have no importance for our purposes.

To estimate Φ_r (defined in (3.11)), let us first note the following estimate.

Lemma 3.20. *We have*

$$\Phi_r(x) \leqslant C(1+|x|)\eta(r).$$

Proof. The proof relies on the same argument as Lemma 3.4. Denote by γ a geodesic segment from e to x, and by Γ_n (for $0 \leqslant n \leqslant |x|$) the set of points whose first projection on γ is the point $\gamma(n)$, at distance n of e. For $y \in \Gamma_n$, one has by Ancona inequalities

$$G_r(e,y)G_r(y,x) \leqslant CG_r(e,\gamma(n))G_r(\gamma(n),y)G_r(y,\gamma(n))G_r(\gamma(n),x) \leqslant CH_r(\gamma(n),y)G_r(e,x).$$

Therefore,

$$\Phi_r(x) = \sum_{n=0}^{|x|}\sum_{y\in\Gamma_n} \frac{G_r(e,y)G_r(y,x)}{G_r(e,x)} \leqslant C\sum_{n=0}^{|x|}\sum_{y\in\Gamma_n} H_r(\gamma(n),y) \leqslant C(|x|+1)\eta(r). \qquad \square$$

To obtain a convergence instead of bounds, we will use a similar argument, but we will need to replace the wild sets Γ_n by a nicer version given by partitions of unity, as in lemma 8.5 of [GL11] (that we recall for the convenience of the reader):

Lemma 3.21. *For K large enough, we can associate to any geodesic segment γ in the Cayley graph of length $2K+1$ centered around e a function $\kappa_\gamma : \Gamma \to [0,1]$ with the following properties:*

(1) *The function κ_γ extends continuously to $\Gamma \cup \partial\Gamma$.*
(2) *Let $\pi_\gamma(y)$ be the set of points on γ that are closest to $y \in \Gamma$. Then $\kappa_\gamma(y) = 0$ if $\pi_\gamma(y)$ contains a point at distance $\geqslant K/4$ of e.*
(3) *Let γ' be any biinfinite geodesic passing through e. Adding the functions κ_γ along the subsegments of γ' of length $2K+1$ one gets the function identically equal to 1. More formally, for all $y \in \Gamma$,*

$$\sum_{n\in\mathbb{Z}} \kappa_{\gamma'(n)^{-1}\gamma'[n-K,n+K]}(\gamma'(n)^{-1}y) = 1. \tag{3.13}$$

Let us now define for $r \in [1,R)$ a function Ψ_r on geodesic segments γ through e, as follows. Let a and b be the endpoints of γ. If $d(e,a) \leqslant K$ or $d(e,b) \leqslant K$, let $\Psi_r(\gamma) = 0$. Otherwise, let

$$\Psi_r(\gamma) = \eta(r)^{-1} \sum_{y\in\Gamma} \kappa_{\gamma[-K,K]}(y) G_r(a,y) G_r(y,b)/G_r(a,b).$$

Consider a geodesic segment γ from e to a point x, and denote by $\sigma^n\gamma$ the shifted segment, i.e., $\gamma(n)^{-1}\gamma$. Then we have

$$\Phi_r(x) = \eta(r) \sum_{n=0}^{|x|} \Psi_r(\sigma^n\gamma) + O(\eta(r)). \tag{3.14}$$

Indeed, by (3.13), when one adds all the quantities $\Psi_r(\sigma^n\gamma)$, one counts every point in the group with a coefficient 1, excepted those whose projection on γ is close to e or x. They contribute to the sum by an amount at most $C\eta(r)$, as explained in the proof of Lemma 3.20.

Lemma 3.22. *The functions Ψ_r are uniformly bounded and Hölder-continuous for $r \in [1,R)$. They converge uniformly when r tends to R.*

By Hölder continuous, we mean that, if two geodesics γ and γ' coincide on a ball of size n around e, then $|\Psi_r(\gamma) - \Psi_r(\gamma')| \leqslant Ce^{-\rho n}$ for some $\rho > 0$.

Proof. This is essentially Lemma 8.6 in [GL11]. The uniform Hölder continuity is proved there and relies uniquely on strong Ancona inequalities. On the other hand, the proof of the convergence when $r \to R$ has to be modified slightly due to the presence of several maximal components.

Since the functions Ψ_r are uniformly Hölder continuous, it is sufficient to show that they converge simply to get uniform convergence. Fix some geodesic segment γ through e, with endpoints a and b at distance at least K of e. We have

$$G_r(a,b)\Psi_r(\gamma) = \frac{1}{\eta(r)} \sum_{y\in\Gamma} \kappa_{\gamma[-K,K]}(y) \frac{G_r(a,y)G_r(y,b)}{G_r(e,y)G_r(y,e)} H_r(e,y) = \int f_r(y)\,\mathrm{d}\nu_r,$$

where $f_r(y) = \kappa_{\gamma[-K,K]}(y) \frac{G_r(a,y)G_r(y,b)}{G_r(e,y)G_r(y,e)}$. This is a function on Γ that extends continuously to $\partial\Gamma$ by the strong Ancona inequalities (and since $\kappa_{\gamma[-K,K]}$ is continuous). Moreover, f_r converges uniformly to a function f_R when r tends to R, by Lemma 3.4. Since ν_r converges weakly by Proposition 3.18, it follows that $\int f_r \,\mathrm{d}\nu_r$ converges. □

Lemma 3.23. *There exists a family of functions h_r on $\overline{\Sigma}$ for $r \in [1, R)$ with the following properties:*

(1) *The functions h_r are Hölder continuous, and they converge in the Hölder topology to a function h_R when $r \to R$.*
(2) *For any $\omega \in \Sigma^*$ of length n,*

$$\Phi_r(\alpha_*(\omega)) = \eta(r) S_n h_r(\omega) + O(\eta(r)). \tag{3.15}$$

We recall that $S_n h_r$ is the Birkhoff sum $\sum_{k=0}^{n-1} h_r \circ \sigma^k$.

Proof. We will need to work with the bilateral shift $\sigma_{\mathbb{Z}}$ on the space $\overline{\Sigma}_{\mathbb{Z}}$ of bilateral paths in the automaton (that may be infinite in zero, one or both directions). We define a function g_r on the set $\Sigma_{\mathbb{Z}}^*$ of finite paths by $g_r(\omega) = 0$ if the length of ω in the future or in the past is less than K, and $g_r(\omega) = \Psi_r(\alpha(\omega))$ otherwise, where $\alpha(\omega)$ is the geodesic segment going through

$$\dots, \alpha(\omega_{-1})^{-1}\alpha(\omega_{-2})^{-1}, \alpha(\omega_{-1})^{-1}, e, \alpha(\omega_0), \alpha(\omega_0)\alpha(\omega_1), \dots .$$

Lemma 3.22 ensures that the functions g_r are Hölder continuous, and that they converge uniformly when r tends to R. By (3.1), they also converge in some Hölder topology. Moreover, they extend to Hölder continuous functions on $\overline{\Sigma}_{\mathbb{Z}}$.

Consider now a finite path ω in Σ^*, of length n. One may consider it as a path in $\Sigma_{\mathbb{Z}}^*$ with empty coordinates for negative time. The equation (3.14) reads

$$\Phi_r(\alpha_*(\omega)) = \eta(r) \sum_{k=0}^{n} g_r(\sigma_{\mathbb{Z}}^k \omega) + O(\eta(r)).$$

This is almost the required property, but the function g_r is defined on the bilateral shift instead of the unilateral shift as desired. This problem is solved using a classical coboundary trick: for any Hölder continuous function g on $\overline{\Sigma}_{\mathbb{Z}}$, there exist two Hölder continuous functions h and u on $\overline{\Sigma}_{\mathbb{Z}}$ (for a smaller Hölder exponent) such that $g = h + u - u \circ \sigma_{\mathbb{Z}}$, and h only depends on positive coordinates. Moreover, h and u depend linearly (and continuously) on g. This is Proposition 1.2 in [PP90]. The proof is given there for subshifts where one only allows infinite paths, but it readily adapts to the situation where finite paths are allowed (or one can reduce to the infinite paths situation by adding two cemeteries, one for the past and one for the future).

Writing $g_r = h_r + u_r - u_r \circ \sigma_{\mathbb{Z}}$ as above, we obtain

$$\Phi_r(\alpha_*(\omega)) = \eta(r) \sum_{k=0}^{n} h_r(\sigma_{\mathbb{Z}}^k \omega) + \eta(r)(u_r(\omega) - u_r(\sigma_{\mathbb{Z}}^{n+1}\omega)) + O(\eta(r)).$$

Since u_r is uniformly bounded, this is the desired decomposition. □

Proof of Proposition 3.17. The sum in (3.10) can be written as $\sum_{x\in\Gamma} H_r(e,x)\Phi_r(x)$. Since α_* induces a bijection between the finite paths in the automaton starting from s_* and the group, this is equal to $\sum H_r(e, \alpha_*(\omega))\Phi_r(\alpha_*(\omega))$, where the sum is restricted to those paths with $\omega_0 \in E_*$.

Consider now, in this sum, the contribution of paths of length n. By definition of the transfer operator $\mathcal{L}_{\varphi_r}$, it is equal to $H_r(e,e)\mathcal{L}^n_{\varphi_r}(1_{[E_*]} \cdot \Phi_r \circ \alpha_*)(\emptyset)$. Using the decomposition (3.15) for $\Phi_r \circ \alpha_*$, we get

$$\sum_{x,y\in\Gamma} G_r(e,y)G_r(y,x)G_r(x,e) = \eta(r) \sum_{n\in\mathbb{N}} \mathcal{L}^n_{\varphi_r}(1_{[E_*]}S_n u_r + O(1))(\emptyset).$$

The contribution of the error term $O(1)$ in this equation is bounded by $\eta(r)\sum \left\|\mathcal{L}^n_{\varphi_r}1\right\| \leqslant C\eta(r)/\left|\mathrm{Pr}(\varphi_r)\right| \leqslant C\eta(r)^2$. It is therefore compatible with the error term in the statement of Proposition 3.17.

Since $\mathcal{L}_{\varphi_r}(u\cdot v\circ\sigma) = v\mathcal{L}_{\varphi_r}u$, we have $\mathcal{L}^n_{\varphi_r}(1_{[E_*]}S_n u_r) = \sum_{k=1}^n \mathcal{L}^k_{\varphi_r}(u_r\mathcal{L}^{n-k}_{\varphi_r}1_{[E_*]})$. Therefore, the previous equation becomes

$$\eta(r)\sum_{n=0}^{\infty}\sum_{k=1}^{n}\mathcal{L}^k_{\varphi_r}(u_r\mathcal{L}^{n-k}_{\varphi_r}1_{[E_*]})(\emptyset)+O(\eta(r)^2) = \eta(r)\sum_{k=1}^{\infty}\mathcal{L}^k_{\varphi_r}\left(u_r\sum_{\ell=0}^{\infty}\mathcal{L}^{\ell}_{\varphi_r}1_{[E_*]}\right)(\emptyset)+O(\eta(r)^2).$$

Using the spectral description of Proposition 3.10, one can write

$$\sum_{\ell=0}^{\infty}\mathcal{L}^{\ell}_{\varphi_r}1_{[E_*]} = \sum_{i=1}^{I} f_{i,r}/\left|\mathrm{Pr}_i(\varphi_r)\right| + O(1),$$

for some Hölder continuous functions $f_{i,r}$ that converge when r tends to R. Again, the $O(1)$ results in an error $O(\eta(r)^2)$ in the final formula. It remains to understand $\sum_{k=1}^{\infty}\mathcal{L}^k_{\varphi_r}(u_r f_{i,r})$. Using again the spectral description of $\mathcal{L}_{\varphi_r}$, one may write it as $\sum_{j=1}^{I} g_{i,j,r}/\left|\mathrm{Pr}_j(\varphi_r)\right|$ for some Hölder continuous functions $g_{i,j,r}$ that depend continuously on r. Finally, we have obtained

$$\sum_{x,y\in\Gamma} G_r(e,y)G_r(y,x)G_r(x,e) = \eta(r)\sum_{i=1}^{I}\sum_{j=1}^{I}\frac{g_{i,j,r}(\emptyset)}{\left|\mathrm{Pr}_i(\varphi_r)\right|\left|\mathrm{Pr}_j(\varphi_r)\right|} + O(\eta(r)^2).$$

By Theorem 3.14 and (3.5), $\eta(r)$ is asymptotic to $\tau_2(r)/\left|\mathrm{Pr}_i(\varphi_r)\right|$ for some continuous function τ_2 that admits a positive limit at $r = R$. Since all the quantities $g_{i,j,r}(\emptyset)$ converge when r tends to R, the proposition follows. □

3.7. **Asymptotics of the Green function.** In this subsection, we prove Theorem 3.1. We rely on the asymptotics for $\sum G_r(e,y)G_r(y,x)G_r(x,e)$ that were obtained in Proposition 3.17 and the differential equation for $G_r(e,e)$.

More precisely, define as in the proof of Corollary 3.3 a function $F(r) = r^2\eta(r)$. It satisfies $F'(r) = 2r\sum_{x,y} G_r(e,y)G_r(y,x)G_r(x,e)$. By Proposition 3.17, $2F'(r)/F(r)^3$ converges to a constant c when r tends to R. By Proposition 3.2, c is nonzero. By integration, it follows that $1/F(r)^2 - 1/F(R)^2 \sim c(R-r)$. Since $F(R) = +\infty$, we get $F(r) \sim c^{-1/2}(R-r)^{-1/2}$. This proves the desired asymptotics of $\partial G_r(e,e)/\partial r$.

Fix now a point $a \in \Gamma$, let us compute the asymptotics of $\partial G_r(e,a)/\partial r$ or, equivalently, of $\sum_x G_r(e,x)G_r(x,a)$. We write this sum as

$$\sum_x H_r(e,x)\frac{G_r(x,a)}{G_r(x,e)} = \eta(r)\int_\Gamma f_r \,\mathrm{d}\nu_r,$$

where $f_r(x) = G_r(x,a)/G_r(x,e)$ and the measure ν_r has been defined in (3.12). The functions f_r extend continuously to $\Gamma \cup \partial\Gamma$ by the strong Ancona inequalities, and converge uniformly when r tends to R, by Lemma 3.4. Since ν_r converges weakly when $r \to R$ by Proposition 3.18, we deduce that $\int f_r \,\mathrm{d}\nu_r$ converges. Therefore, the asymptotics of $\partial G_r(e,a)/\partial r$ follow from those of $\eta(r)$. $\square$

4. Asymptotics of transition probabilities

Theorem 1.1 follows directly from the asymptotics of the Green function proved in Theorem 3.1 and from Theorem 9.1 in [GL11]: this theorem shows that, for *symmetric* measures, one can read the behavior of transition probabilities from the behavior of the Green function.

Let us explain quickly why symmetry matters. The Green function is $\sum r^n p_n(x,y)$, its derivative is $\sum nr^{n-1}p_n(x,y)$. If $\partial G_r(x,y)/\partial r \sim C(R-r)^{-1/2}$, it follows from Karamata's tauberian theorem that

$$\sum_{k=1}^n kR^k p_k(x,y) \sim C' n^{1/2}.$$

This is a local limit theorem in Cesaro average. If $R^n p_n(x,y)$ were monotone, the desired asymptotics of $p_n(x,y)$ would follow readily. Symmetry is used to obtain almost monotonicity: up to an exponentially small error (that does not matter in the estimates), $R^n p_n(x,y)$ is indeed decreasing when the random walk is aperiodic and the measure is symmetric. This is a consequence of spectral results for the (self-adjoint) Markov operator associated to the random walk.

From Theorem 1.1, one can also derive asymptotics for the first return probabilities. We describe the result in the aperiodic case, the periodic one is handled similarly by looking at μ^2.

Proposition 4.1. *Consider a probability measure μ on a countable group Γ such that the associated transition probabilities satisfy $p_n(x,y) \sim C(x,y)R^{-n}n^{-\beta}$ for some $R \geqslant 1$ and $\beta > 1$. Let $f_n(x,y)$ be the first visit probabilities from x to y at time n, i.e.,*

$$f_n(x,y) = \mathbb{P}_x(X_1,\dots,X_{n-1} \neq y, X_n = y).$$

Then $f_n(x,y) \sim C'(x,y)R^{-n}n^{-\beta}$ for some constants $C'(x,y)$.

For the proof, we will mainly rely on the following theorem ([CNW73, Theorem 1]):

Theorem 4.2. *Consider a function $A(z) = \sum_{n=0}^\infty a_n z^n$ with $a_n \geqslant 0$ and $a_n \sim cR^{-n}n^{-\beta}$ with $\beta > 1$. Consider also a function Φ which is analytic on a neighborhood of $\{f(z) : |z| \leqslant R\}$. Then the coefficients b_n of the series expansion $\Phi(A(z)) = \sum b_n z^n$ satisfy*

$$b_n \sim a_n \Phi'\left(\sum a_n R^n\right).$$

Proof of Proposition 4.1. Decomposing a path from e to itself into successive excursions, one gets the renewal equation

$$\sum_{n=0}^{\infty} p_n(e,e)z^n = \frac{1}{1-\sum_{n=1}^{\infty} f_n(e,e)z^n}. \tag{4.1}$$

Since $\sum p_n(e,e)R^n = G_R(e,e) < \infty$, one deduces $\sum f_n(e,e)R^n < 1$. Therefore, $1 - \sum f_n(e,e)z^n$ does not vanish for $|z| \leqslant R$, and $\sum p_n(e,e)z^n$ is well defined and nonzero for any such z. From (4.1), one gets

$$\sum f_n(e,e)z^n = 1 - 1/\sum p_n(e,e)z^n = \Phi(\sum p_n(e,e)z^n),$$

where we set $\Phi(t) = 1 - 1/t$. Since $p_n(e,e) \sim CR^{-n}n^{-\beta}$, we may apply Theorem 4.2 to obtain $f_n(e,e) \sim C'R^{-n}n^{-\beta}$.

For $x \neq y$, one has $\sum p_n(x,y)z^n = (\sum f_n(x,y)z^n) \cdot (\sum p_n(e,e)z^n)$. The functions $\sum p_n(x,y)z^n$ and $1/\sum p_n(e,e)z^n$ both have coefficients that are asymptotic to a constant times $R^{-n}n^{-\beta}$. Since the set of all functions with this property is closed under multiplication (see [CNW73, Lemma 1]), we get $f_n(x,y) \sim C'(x,y)R^{-n}n^{-\beta}$ as desired. □

Appendix A. Ancona inequalities for surface groups

In this appendix, we prove Ancona inequalities for surface groups without any symmetry assumption on the measure:

Theorem A.1. *Let Γ be a cocompact Fuchsian group, i.e., a cocompact discrete subgroup of* $\mathrm{PSL}(2,\mathbb{R})$*. Let μ be an admissible finitely supported probability measure on Γ. Then it satisfies strong uniform Ancona inequalities.*

This theorem has several corollaries:

Corollary A.2. *Under the assumptions of the theorem, the Martin boundary for R-harmonic functions coincides with the geometric boundary of the group, i.e., the unit circle S^1.*

Corollary A.3. *Under the assumptions of the theorem, for any $x,y \in \Gamma$, there exists $C(x,y) > 0$ such that $\sum_{k=1}^{n} kR^k p_k(x,y) \sim C(x,y)n^{1/2}$.*

The first corollary is a classical consequence of Ancona inequalities. For the second corollary, we rely on Section 3 (or on [GL11], since the Cannon automaton is transitive) to deduce that the Green function satisfies $\partial G_r(x,y)/\partial r \sim C(x,y)/\sqrt{R-r}$ when r tends to R. Using Karamata's tauberian theorem, this readily implies the statement of the corollary (see the arguments in Section 4). Note that we are unable to deduce the true local limit theorem from this estimate since we do not know if $R^n p_n(x,y)$ is decreasing, or sufficiently well approximated by a decreasing sequence, as in the symmetric situation.

The strong uniform Ancona inequalities of Theorem A.1 are a consequence of estimates for the weight of paths avoiding a ball, given in the following proposition, and of Lemma 2.7.

Proposition A.4. *Let Γ be a cocompact Fuchsian group in* $\mathrm{PSL}(2,\mathbb{R})$*. Let μ be an admissible finitely supported probability measure on Γ. For any $K > 0$, there exists $n_0 > 0$ such that, for any $n \geqslant n_0$, for any points x,y,z on a geodesic segment (in this order) with $d(x,y) \in [n, 100n]$ and $d(y,z) \in [n, 100n]$,*

$$G_R(x,z;B(y,n)^c) \leqslant K^{-n}.$$

The rest of this section is devoted to the proof of the proposition. The main tool in this proof is superadditivity (as in Lemma 2.6, that relies on Lemma 2.5). This time, it is in the form of Kingman's subadditive ergodic theorem, or rather a bilateral version of this theorem that we now give.

Theorem A.5. *Let $T : \Omega \to \Omega$ be an ergodic probability preserving invertible automorphism of a probability space $(\Omega, \mathbb{P})$. Consider for each bounded interval $I \subset \mathbb{Z}$ an integrable function $\Phi_I : \Omega \to \mathbb{R}$ with the following properties:*

(1) *If I is the disjoint union of two intervals I_1 and I_2, then $\Phi_I(\omega) \leqslant \Phi_{I_1}(\omega) + \Phi_{I_2}(\omega)$.*
(2) *One has $\Phi_{[m,n]}(\omega) = \Phi_{[m-1,n-1]}(T\omega)$.*
(3) *The quantity $n^{-1} \int \Phi_{[0,n)}(\omega) \, \mathrm{d}\mathbb{P}(\omega)$ is bounded from below.*

Then, for almost every ω, the quantity $(m+n)^{-1}\Phi_{[-m,n)}(\omega)$ converges when $m + n \to \infty$ (and $m, n \geqslant 0$) towards $\inf n^{-1} \int \Phi_{[0,n)}(\omega) \, \mathrm{d}\mathbb{P}(\omega) = \lim n^{-1} \int \Phi_{[0,n)}(\omega) \, \mathrm{d}\mathbb{P}(\omega)$.

Proof. Let $\Psi_n(\omega) = \Phi_{[0,n)}(\omega)$. The assumptions give, for any $m, n \geqslant 0$,

$$\begin{aligned}\Psi_{m+n}(\omega) = \Phi_{[0,m+n)}(\omega) &\leqslant \Phi_{[0,n)}(\omega) + \Phi_{[n,n+m)}(\omega) = \Phi_{[0,n)}(\omega) + \Phi_{[0,m)}(T^n\omega) \\ &= \Psi_n(\omega) + \Psi_m(T^n\omega).\end{aligned}$$

This shows that Ψ_n is a subadditive cocycle in the usual sense of Kingman's ergodic theorem (see for instance [Kre85, Theorem I.5.3]). Therefore, $n^{-1}\Psi_n$ converges almost surely and in L^1 to the limit $c = \inf n^{-1} \int \Phi_{[0,n)}(\omega) \, \mathrm{d}\mathbb{P}(\omega) = \lim n^{-1} \int \Phi_{[0,n)}(\omega) \, \mathrm{d}\mathbb{P}(\omega)$.

In the same way, $\Phi_{[-n,-1]}(\omega)$ is a subadditive cocycle for the transformation T^{-1}. Hence, $n^{-1}\Phi_{[-n,-1]}(\omega)$ converges almost surely to $\lim n^{-1} \int \Phi_{[-n,-1]}(\omega) \, \mathrm{d}\mathbb{P}(\omega)$. Changing variables by $\omega' = T^{-n}\omega$, this integral is equal to $\int \Phi_{[0,n)}(\omega') \, \mathrm{d}\mathbb{P}(\omega')$. Therefore, the limit is again c.

Consider now a generic point ω, and $m, n \geqslant 0$. We want to show that if $m + n$ is large then $(m+n)^{-1}\Phi_{[-m,n)}(\omega)$ is close to c. We will do so if m is large, the case n large is handled similarly. Let $\varepsilon > 0$ be small. Since $j^{-1}\Phi_{[0,j)}(\omega')$ converges almost everywhere to c, it converges uniformly on a set A of measure arbitrarily close to 1. In particular, there exists N such that, for all $j \geqslant N$ and for all $\omega' \in A$, one has $j^{-1}\Phi_{[0,j)}(\omega') \in [c - \varepsilon, c + \varepsilon]$. Since ω is generic and $\mathbb{P}(A)$ is very close to 1, the orbit of ω spends a very large proportion of its time in A. In particular, one may find for every large enough m an integer $k \in [m + \varepsilon m, m + 2\varepsilon m]$ such that $T^{-k}\omega \in A$. We get for this k (and for any $n \geqslant 0$)

$$\Phi_{[0,k+n)}(T^{-k}\omega) \leqslant \Phi_{[0,k-m)}(T^{-k}\omega) + \Phi_{[k-m,k+n)}(T^{-k}\omega) = \Phi_{[0,k-m)}(T^{-k}\omega) + \Phi_{[-m,n)}(\omega).$$

If m is large enough, then $k - m \geqslant \varepsilon m$ is larger than N. Since $T^{-k}\omega$ belongs to A, we obtain

$$\Phi_{[-m,n)}(\omega) \geqslant (k+n)(c-\varepsilon) - (k-m)(c+\varepsilon) = (m+n)(c + O(\varepsilon)).$$

This shows that $\liminf (m+n)^{-1}\Phi_{[-m,n)}(\omega) \geqslant c$. On the other hand,

$$\Phi_{[-m,n)}(\omega) \leqslant \Phi_{[-m,-1]}(\omega) + \Phi_{[0,n)}(\omega) = m(c + o(1)) + n(c + o(1)).$$

Therefore, $\limsup (m+n)^{-1}\Phi_{[-m,n)}(\omega) \leqslant c$. This concludes the proof. □

Let us now start the proof of Proposition A.4. We can assume without loss of generality that $y = e$. Let $\ell > 0$ be large (it will not depend on n), we will construct ℓ suitable barriers

$A_1, \dots, A_\ell$ between x and z such that

$$\sum_{a \in A_i, b \in A_{i+1}} G_R(a,b)^2 \leqslant e^{-\rho n} \tag{A.1}$$

and

$$\sum_{a \in A_1} G_R(e,a)^2 \leqslant 1, \quad \sum_{a \in A_\ell} G_R(a,e)^2 \leqslant 1 \tag{A.2}$$

for some $\rho > 0$ that does not depend on ℓ, x or z. From the last equation, we obtain $\sum_{a \in A_1} G_R(x,a)^2 \leqslant C^n$ thanks to Harnack inequalities (and since $d(x,e) \leqslant 100n$), and $\sum_{a \in A_\ell} G_R(a,z)^2 \leqslant C^n$. Arguing as in the beginning of the proof of Lemma 2.6, we define operators L_i from $\ell^2(A_{i+1})$ to $\ell^2(A_i)$. They satisfy $\|L_0\| \leqslant C^{n/2}$, $\|L_{\ell+1}\| \leqslant C^{n/2}$ and $\|L_i\| \leqslant e^{-\rho n/2}$ for $1 \leqslant i \leqslant \ell$. Therefore,

$$G_R(x,z; B(e,n)^c) \leqslant \prod \|L_i\| \leqslant C^n e^{-(\ell-1)\rho n/2}.$$

Taking ℓ large, we can ensure that this is bounded by K^{-n} as desired, for any $K > 0$.

The key point of the argument is the construction of the barriers. The problem with the argument in Lemma 2.6 is that we only have a control on $G_R(a,b)G_R(b,a)$ coming from Lemma 2.5, not $G_R(a,b)^2$. The idea is that those controls would be equivalent if $G_R(a,b)$ and $G_R(b,a)$ were of the same order of magnitude. For symmetric measures, this is always the case. For non-symmetric measures, we will be able to enforce it by constructing the barriers using another, *symmetric*, random walk, and use Kingman subadditive ergodic theorem to show that for typical points both $G_R(a,b)$ and $G_R(b,a)$ grow at the same speed. It will then follow from Lemma 2.5 that they are both exponentially small.

Let us stress that this kind of argument can not work for *all* points. For instance, consider in the free group on two generators a and b a random walk that goes towards a with probability $1-3\varepsilon$, and towards a^{-1}, b and b^{-1} with probability ε, for some small enough ε. It is easy to check that $G_R(e,a^n)$ is exponentially large (while $G_R(a^n,e)$ is exponentially small). In particular, $\sum_{x \in \mathbb{S}_n} G_R(e,x)^2$ grows exponentially fast, but this growth is due to a rather small number of points. The barriers we construct have to avoid those points.

We turn to details. The barriers we will construct will not depend on the points x and z (but the order in which they will be encountered will depend on those points, of course). Since Γ is a cocompact discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, it acts on the hyperbolic disk $\mathbb{H}^2$. If O is a suitably chosen reference point in this disk, the points γO (for $\gamma \in \Gamma$) are pairwise disjoint, hence Γ can be identified with ΓO. Moreover, this identification is a quasi-isometry between Γ (with the word distance coming from its Cayley graph) and $\mathbb{H}^2$. In particular, the geometric boundary of Γ is identified with $S^1 = \partial\mathbb{H}^2$. We can assume that O is the center of the hyperbolic disk.

Let us fix an admissible *symmetric* measure ν on Γ, supported on the set of generators, and let us consider the corresponding random walk. We claim that the following lemma holds. Here and henceforth, G_R always denotes the Green function associated to the original measure μ.

Lemma A.6. *There exist $\rho > 0$ and $v > 0$ with the following properties.*

(1) *For almost every trajectory X_k of the random walk given by ν, $d(X_k, e) \sim kv$.*

(2) *For almost every trajectory X_k, for all large enough k, $G_R(e, X_k) \leqslant e^{-\rho k}$ and $G_R(X_k, e) \leqslant e^{-\rho k}$.*
(3) *For almost every pair of independent trajectories X_k and Y_k, for all large enough k and k', $G_R(X_k, Y_{k'}) \leqslant e^{-\rho(k+k')}$.*

Let us admit the lemma for the moment. We choose $2\ell + 2$ points in S^1 that are evenly spaced, and $2\ell + 2$ small intervals I_i around those points. The Poisson boundary of the random walk given by ν is S^1, and the hitting measure has full support. Therefore, there is positive probability to hit the boundary in any of the intervals I_i. Let us choose for each i a trajectory $X_k^{(i)}$ of the random walk that ends up in I_i. We will also require each of those trajectories to be typical, so that they satisfy the conclusions of Lemma A.6.

Since the trajectories $X_k^{(i)}$ converge to different points on the boundaries, they are disjoint outside of a large enough compact set. Let $\Gamma_i(n)$ be the set of points $X_k^{(i)}$ that are at distance at least n of e, and let $B_i(n)$ be a thickening of $\Gamma_i(n)$, i.e., $B_i(n) = \bigcup_{a\in\Gamma_i(n)} B(a, C_0)$ for some large constant C_0. If n is large enough, the sets $(B_i(n))_{i\leqslant 2\ell+2}$ are mutually disjoint. Since $d(X_k^{(i)}, e) \sim vk$, the set $\Gamma_i(n)$ only contains points among the $X_k^{(i)}$ with $k \geqslant n/(2v)$. Therefore,

$$\sum_{a\in\Gamma_i(n)} G_R(e, a)^2 \leqslant \sum_{k=n/(2v)}^{\infty} G_R(e, X_k^{(i)})^2 \leqslant \sum_{k=v/(2v)}^{\infty} e^{-2\rho k} \leqslant Ce^{-\rho n/v}.$$

Since any point in the thickening $B_i(n)$ is a bounded distance away from a point in $\Gamma_i(n)$, a similar estimate holds with $B_i(n)$ instead of $\Gamma_i(n)$ thanks to Harnack inequalities. Arguing in the same way for the other inequalities, we obtain

$$\begin{gathered}\sum_{a\in B_i(n)} G_R(e, a)^2 \leqslant Ce^{-\rho n/v}, \quad \sum_{a\in B_i(n)} G_R(a, e)^2 \leqslant Ce^{-\rho n/v}, \\ \sum_{a\in B_i(n), b\in B_j(n)} G_R(a, b)^2 \leqslant Ce^{-2\rho n/v}.\end{gathered} \tag{A.3}$$

Consider now two points x and z at distance at least n of e such that e is on a geodesic segment from x to z. The Cayley geodesic from x to z is a quasi-geodesic in hyperbolic space, that remains in a bounded size neighborhood of a true hyperbolic geodesic (and this geodesic passes close to O). Avoiding a ball around O in $\mathbb{H}^2$, one can go from x to z in two directions around this ball, clockwise or counterclockwise. Since the limit intervals I_i are evenly spaced, it follows that, in any of those directions, one meets successively at least ℓ sets $B_i(n)$ (discarding if necessary the two sets that contain x and z). Denote by A_1 the union of the two sets $B_i(n)$ that are closest to x (ignoring the single set that might contain x), then A_2 the union of the next two ones, and so on. We get barriers $A_1, \dots, A_\ell$ between x and z as desired. Moreover, the estimates (A.3) show that those barriers satisfy (A.1) and (A.2) if n is large enough (for a different value of ρ). This concludes the proof of Proposition A.4, modulo Lemma A.6. □

Proof of Lemma A.6. We will use Kingman's theorem on the space $\Omega = \Gamma^{\mathbb{Z}}$ with the product measure $\nu^{\otimes\mathbb{Z}}$. In other words, an element $\omega \in \Omega$ is a sequence of elements ω_i of Γ that are drawn independently according to ν. Such an ω can be viewed as the increments of a random walk distributed according to ν: let $X_n(\omega) = \omega_0 \cdots \omega_{n-1}$ for $n \geqslant 0$ and $X_n(\omega) = \omega_{-1}^{-1} \cdots \omega_n^{-1}$ for $n < 0$, so that $X_0 = e$ and $X_{n+1} = X_n \omega_n$. Let T be the left shift on Ω, it is ergodic, preserves the measure, and $X_n(T\omega) = \omega_0^{-1} X_{n+1}(\omega)$.

We define a subadditive cocycle $\Phi_{[m,n)}(\omega) = -\log F_R(X_m(\omega), X_n(\omega))$, where $F_R(x,y) = G_R(x,y)/G_R(e,e)$ is the first entrance Green function defined in Subsection 2.1. This function satisfies $F_R(x,y)F_R(y,z) \leqslant F_R(x,z)$ by (2.3), hence $\Phi_{[m,n)}$ is subadditive. By Harnack inequalities, $\left|\Phi_{[m,n)}(\omega)\right| \leqslant C\,|n-m|$. Therefore, the integrability assumptions of Theorem A.5 are satisfied. We deduce that $(m+n)^{-1} \log F_R(X_{-m}, X_n)$ converges almost surely to $c = \lim n^{-1} \int \log F_R(e, X_n(\omega))\,\mathrm{d}\mathbb{P}(\omega)$. We can also apply Kingman's theorem to the cocycle $\tilde{\Phi}_{[m,n)}(\omega) = -\log F_R(X_n(\omega), X_m(\omega))$, to get that $(m+n)^{-1} \log F_R(X_n, X_{-m})$ almost surely converges, to $c' = \lim n^{-1} \int \log F_R(X_n(\omega), e)\,\mathrm{d}\mathbb{P}(\omega)$. We have $F_R(X_n, e) = F_R(e, X_n^{-1})$. Since X_n^{-1} is distributed like X_n by symmetry of the random walk, this gives $\int \log F_R(X_n(\omega), e)\,\mathrm{d}\mathbb{P}(\omega) = \int \log F_R(e, X_n(\omega))\,\mathrm{d}\mathbb{P}(\omega)$. Dividing by n and letting n tends to infinity gives $c = c'$.

The quantities $(m+n)^{-1} \log F_R(X_{-m}, X_n)$ and $(m+n)^{-1} \log F_R(X_n, X_{-m})$ both converge to c. Taking $m = 0$, we get in particular that $n^{-1} \log F_R(e, X_n)$ and $n^{-1} \log F_R(X_n, e)$ almost surely converge to c.

We will now prove that c is strictly negative. There exists a real number $h \geqslant 0$ (the entropy of the random walk) such that the random walk at time n is essentially supported by e^{hn} points, with a probability e^{-hn} to reach each of those points. More precisely (see for instance [Fur02, Theorem 2.28]), for any $\varepsilon > 0$, if n is large enough, there exists a subset E_n of Ω, with probability at least $3/4$, such that for any $\omega \in E_n$ one has

$$\mathbb{P}\{\omega' : X_n(\omega') = X_n(\omega)\} \in [e^{(-h-\varepsilon)n}, e^{(-h+\varepsilon)n}].$$

Since $\log F_R(e, X_n)$ and $\log F_R(X_n, e)$ almost surely converge to c, we can also assume (shrinking E_n a little bit) that for any $\omega \in E_n$ one has $F_R(e, X_n) \geqslant e^{(c-\varepsilon)n}$ and $F_R(X_n, e) \geqslant e^{(c-\varepsilon)n}$. Let $\tilde{E}_n \subset \Gamma$ be the set of points $X_n(\omega)$ for $\omega \in E_n$. It has cardinality at least $Ce^{(h-\varepsilon)n}$, it is contained in $B(e,n)$ (since the steps of the random walk have length at most 1 by definition), and for any $x \in \tilde{E}_n$ one has $F_R(e,x)F_R(x,e) \geqslant e^{2(c-\varepsilon)n}$. Therefore,

$$\sum_{x \in B(e,n)} F_R(e,x)F_R(x,e) \geqslant \sum_{x \in \tilde{E}_n} F_R(e,x)F_R(x,e) \geqslant \operatorname{Card} \tilde{E}_n \cdot e^{2(c-\varepsilon)n} \geqslant Ce^{(h-\varepsilon)n} e^{2(c-\varepsilon)n}.$$

By Lemma 2.5, the sum $\sum_{x \in \mathbb{S}_k} F_R(e,x)F_R(x,e)$ is uniformly bounded (since $F_R(x,y) = G_R(x,y)/G_R(e,e)$). Therefore, $\sum_{x \in B(e,n)} F_R(e,x)F_R(x,e) \leqslant Cn$. We deduce that $(h-\varepsilon) + 2(c-\varepsilon)$ is nonpositive. Finally, letting ε tend to 0, we get $c \leqslant -h/2$. Since entropy is nonzero in non-amenable groups (see for instance [Fur02, Proposition 2.35]), we get $c < 0$ as desired.

Let us now prove the estimates of the lemma. The first item (positive escape rate) is classical and follows from Kingman's theorem for the existence of the escape rate, and from the inequality $h \leqslant v\zeta$ for its positivity (where ζ is the exponential growth rate of the cardinality of balls), see [Fur02, Proposition 2.32]. For the second item, consider a typical trajectory X_k of the random walk. Since $\log F_R(e, X_k) \sim ck$ with $c < 0$, we deduce that for large enough k one has $F_R(e, X_k) \leqslant e^{ck/2}$. Since $G_R(x,y) = F_R(x,y)G_R(e,e)$, the exponential decay of $G_R(e, X_k)$ follows. The decay of $G_R(X_k, e)$ is handled in the same way. Finally, consider two independent trajectories X_k and Y_k of the random walk. Define (for $k > 0$) $X_{-k} = Y_k$. By symmetry of ν, $(X_k)_{k\in\mathbb{Z}}$ is a typical trajectory for the bilateral random walk. Applying Theorem A.5, we deduce that $\log F_R(X_{-m}, X_n) \sim (m+n)c$, i.e., $\log F_R(Y_m, X_n) \sim (m+n)c$ when $m+n$ tends to infinity. This is the desired exponential decay. $\square$

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